

Risk and Macro Finance Working Paper Series

No. 2014-03

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Consumption and Portfolio Choice under Loss Aversion and Endogenous Updating of the Reference Level*

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November 5, 2014

This paper explicitly derives the optimal dynamic consumption and portfolio choice of a loss averse agent who endogenously updates his reference level. His optimal choice seeks protection against consumption losses due to downside financial shocks. This induces a (soft) guarantee on consumption and is due to loss aversion. Furthermore, his optimal consumption choice gradually adjusts to financial shocks. This resembles the payout streams of financial plans that respond sluggishly, smoothing investment returns to reduce payout volatility, and is due to endogenous updating. The welfare losses associated with various suboptimal consumption and portfolio strategies are also evaluated. They can be substantial.

Keywords: optimal consumption choice, optimal portfolio choice, loss aversion, reference level, habit formation, guarantee, smoothing

JEL Code(s): D81, D91, G02, G11

*We are very grateful to Lans Bovenberg, Louis Eeckhoudt, Michel Vellekoop and seminar and conference participants at Tilburg University, the University of Amsterdam, the Netspar International Pension Workshop, the Foundations for Utility and Risk (FUR) Meeting 2014, and the Insurance: Mathematics and Economics Meeting 2014. This research was funded in part by the Netherlands Organization for Scientific Research (NWO) under grant NWO Vidi 2009 (van Bilsen, Laeven).

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1 Introduction

The pension fund industry has grown dramatically over the past four decades: U.S. total retirement assets rose from 369 billion dollars in 1974 to 23 trillion dollars in 2013 (ICI, 2014). During the same period, we have seen in particular a pronounced increase in retirement saving through personal retirement accounts, such as IRAs and DC plans (Poterba, Venti, and Wise, 2009). More specifically, the percentage of U.S. total retirement assets accounted for by IRAs and DC plans grew from about 18% in 1974 to about 54% in 2013 (ICI, 2014). These figures highlight the importance of adequate individual consumption, savings and investment decisions over the life cycle, and of the design of such individual financial plans.

Since the seminal works of Merton (1969, 1971) and Samuelson (1969), a considerable number of authors have studied optimal consumption and portfolio choice over the life cycle in a wide variety of settings. Standard life cycle models assume that preferences are represented by expected utility with constant relative risk aversion (CRRA); see, e.g., Wachter (2002), Cocco, Gomes, and Maenhout (2005), Liu (2007), Gomes, Kotlikoff, and Viceira (2008), to name just a few. With such standard preferences (and without constraints), the optimal log consumption choice is a *linear* function of the log state price density (see, e.g., Karatzas and Shreve, 1998, p. 103). Furthermore, under such standard preferences, financial shocks are *directly* absorbed into the optimal log consumption choice: a CRRA agent chooses to instantaneously adjust consumption to financial shocks.

These predictions of standard life cycle models stand in sharp contrast to *actual* income streams generated by financial and insurance products. Financial fiduciaries have developed a variety of features, options and guarantees so as to make base financial products more attractive for individuals (see, e.g., van Rooij, Kool, and Prast, 2007; Antolín, Payet, Whitehouse, and Yermo, 2011; Bodie and Taqqu, 2011). These include guaranteed minimum income benefits, guaranteed minimum withdrawal benefits and minimum rate of return guarantees. In addition, many actively traded financial derivative securities have a nonlinear payoff structure, and provide some degree of protection against downside risk. The popularity of these products contradicts the linearity of the standard consumption rule.

Furthermore, a substantial body of literature (see, e.g., Sundaresan, 1989; Constantinides, 1990) argues that agents become accustomed to a certain level of consumption. This strand of the literature suggests that agents evaluate and adjust consumption relative to a reference (or a habit) level. The empirical literature (see, e.g., Lupton, 2003) provides evidence of habit persistence in consumption, with consumption being smooth relative to wealth. Moreover, financial fiduciaries (such as life insurers and pension funds) increasingly offer plans with payout streams that are not directly but only sluggishly linked to the performance of the underlying investment portfolio.¹ There have been numerous attempts to reconcile theory and practice

¹In many European countries, but also in the US and Japan, the importance of participating (or with profits) annuities is growing (see, e.g., Guillén, Jørgensen, and Nielsen, 2006; Maurer, Mitchell, and Rogalla, 2010). A key characteristic of participating annuities is that investment returns are smoothed so as to reduce payout volatility. For example, in the Netherlands, pension funds are allowed to gradually absorb financial shocks into

of life cycle consumption and portfolio choice. However, to the best of our knowledge, the literature has not yet been able to provide a fully satisfactory answer that accommodates these features — nonlinearity of the consumption rule and smoothing of financial shocks — all together.

In this paper, we explore consumption and portfolio choice under reference-dependent preferences. More specifically, we analyze optimal consumption and portfolio choice under the utility (or value) function of prospect theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992) and adopt an endogenous updating mechanism for the dynamics of the reference level.² The consumption and portfolio choice model we consider is able to generate both a nonlinear consumption rule and smoothing of financial shocks in an integrated framework. The optimal choice seeks protection against consumption losses due to financial shocks inducing a (“soft”) guarantee on consumption. Furthermore, the optimal consumption choice exhibits sluggish response to financial shocks.

Following prospect theory, we assume that the agent’s instantaneous utility function is represented by the two-part power utility function. This utility function incorporates several behavioral properties, such as reference dependence (i.e., the carriers of utility are gains and losses rather than absolute levels of consumption), loss aversion (i.e., losses hurt more than gains satisfy), and diminishing sensitivity (i.e., the impact of a marginal change in consumption decreases as the agent moves further away from the reference level).^{3,4} Diminishing sensitivity implies a convex utility function below the reference level.⁵ The empirical literature is, however, inconclusive as to whether the utility function is convex in the loss domain; see, e.g., Abdellaoui, Vossman, and Weber (2005).⁶ Therefore, the current paper considers not only the case of a convex utility function in the loss domain, but also the case of a concave utility function in the loss domain.⁷

Our main results can be summarized as follows. First, we demonstrate that the agent optimally chooses to divide the states of the economy into two categories: insured states (i.e., good to intermediate economic scenarios or, equivalently, low to intermediate state prices) and *uninsured* states (i.e., bad economic scenarios or high state prices). In insured states, consumption is guaranteed to be larger than the reference level, while in uninsured states, consumption is smaller than the reference level. If consumption is larger (smaller) than the

pension entitlements. Also, life insurers use special smoothing techniques in an attempt to stabilize payouts.

²We abstract away from (subjective) probability weighting.

³Kőszegi and Rabin (2006, 2007) develop a class of reference-dependent preferences with endogenous updating (and without probability weighting). See Section 3 for further details about the connection between the class of Kőszegi and Rabin (2006, 2007) and our model.

⁴According to Wakker (2010, p. 242), “reference dependence, in combination with loss aversion, is one of the most pronounced empirical phenomena in decision under risk and uncertainty.”

⁵We note that, in our context, a convex utility function implies risk-seeking behavior.

⁶Etchart-Vincent (2004) investigated the sensitivity of the utility function to the magnitude of the underlying payoffs. She found that a larger proportion of the subjects exhibited concavity when facing large losses than when facing small losses.

⁷The literature also provides some support for the idea that agents exhibit an inverted S-shaped utility function in the loss domain. For example, Laughhunn, Payne, and Crum (1980) found that a large proportion of the subjects (64%) switched from risk-seeking to risk-averse behavior when facing ruinous losses.

reference level, then the agent experiences a gain (loss). Because of loss aversion, the agent has a strong preference to maintain consumption above the reference level, but when the state of the economy is really bad, the (soft) guarantee on consumption can no longer be maintained. More specifically, the optimal consumption profile (i.e., the optimal consumption choice as a function of the log state price density) displays a 90° rotated S-shaped pattern.⁸ We show that when the agent becomes more afraid of incurring losses, the probability of consumption falling below the reference level decreases. At the same time, the agent must give up some upward potential in order to finance this more conservative consumption profile.

Second, under our preference assumptions, the optimal consumption choice gradually adjusts to financial shocks. [Kahneman and Tversky \(1979\)](#) argue that the status quo, an expectation or an aspiration level can serve as a reference level, but do not specify how the reference level is formed and updated over time. Following the internal habit formation literature (see, e.g., [Constantinides, 1990](#)), we assume that the reference level depends on the agent’s own past consumption choices. More specifically, we assume that the reference level can be decomposed into two components: a stochastic and a deterministic component.⁹ The stochastic component is given by an exponentially weighted average of the agent’s own past consumption choices. The specification of the reference level is motivated by the idea that agents become accustomed to a certain level of consumption. A main implication of the consumption and portfolio choice model we consider is that after a financial shock, optimal consumption adjustment is sluggish (at least in the short run). That is, a current financial shock has a larger impact on consumption in the distant future than on consumption in the near future. Part of the financial shock will be directly reflected into gains and losses, another part will smoothly enter through the reference level, which is endogenously updated over time.

Third, the optimal portfolio profile displays a U-shaped pattern: the total dollar amount invested in risk-bearing assets will be lower in intermediate economic scenarios than in good or bad economic scenarios. As a by-product of interest in its own right, the agent implements a life cycle investment strategy, even without taking human capital into account.¹⁰ Since the agent has less time to absorb financial shocks as he grows older, the equity risk exposure, on average, decreases over the life cycle.

Finally, to investigate the impact of implementing suboptimal consumption and portfolio strategies on the agent’s welfare, we conduct a welfare analysis. We compute the welfare losses (in terms of the relative decline in certainty equivalent consumption) associated with

⁸The exact behavior of the agent below the reference level depends on the shape of utility function in the loss domain.

⁹The reference level is characterized by three parameters: the initial reference level, an endogeneity parameter (which measures the degree of endogeneity) and a depreciation parameter (which measures the rate at which the agent depreciates the reference level).

¹⁰Under CRRA utility, the agent has a constant equity risk exposure if the investment opportunity set is assumed to be constant. [Bodie, Merton, and Samuelson \(1992\)](#) give a justification for adopting a life cycle investment strategy based on human capital considerations. If human capital is risk-free, then agents implicitly hold a risk-free asset. To offset this implicit risk-free asset holding, financial wealth should be tilted toward risky assets. As the share of human capital in total wealth decreases from one to zero during the working period, the optimal proportion of financial wealth invested in risk-free assets increases over the life cycle.

implementing suboptimal consumption and portfolio strategies. Because of the endogeneity of the reference level, this requires a non-standard computation of certainty equivalents. The results indicate that welfare losses can be substantial. Particularly, for our realistic parameter values, we find that the welfare loss associated with implementing the classical Merton strategy (see [Merton, 1969](#)) can be as large as 40%. We also compute the welfare losses of suboptimal behavior due to incorrect assumptions on the underlying agent’s preference parameters. We find that consumption and portfolio strategies based on incorrectly assuming a constant exogenous reference level (or only a very limited degree of endogeneity), thus implying no (or only very limited) smoothing of financial shocks, substantially reduce welfare.

To solve the consumption and portfolio choice model, we first apply the solution technique of [Schroder and Skiadas \(2002\)](#). This method enables us to transform the consumption and portfolio choice model with endogenous updating into a *dual* consumption and portfolio choice model without endogenous updating. The dual utility function is time-additive and separable. This fact facilitates the derivation of the optimal consumption and portfolio choice. Next, we solve the dual problem by using convex-duality (or martingale) techniques, and by using techniques proposed by [Basak and Shapiro \(2001\)](#) and [Berkelaar, Kouwenberg, and Post \(2004\)](#) to deal with pseudo-concavity and non-differentiability aspects of the problem. We adapt the latter techniques to our setting with intertemporal consumption. Upon transforming our solutions under the dual model back into the primal model, we finally arrive at explicit closed-form solutions to our initial problem under consideration.

The literature on optimal consumption *and* portfolio choice under prospect theory type preferences is scarce. [Berkelaar et al. \(2004\)](#) examine analytically optimal portfolio choice under the two-part power utility function. Their paper differs from ours in at least two main respects. First and foremost, we assume that the agent is concerned not with terminal wealth, but with intertemporal consumption. This allows us to examine how the agent’s consumption strategy evolves as time proceeds and risk resolves, which is our prime focus. Second, in this setting with intertemporal consumption, we allow the agent to not just stochastically but also endogenously update his reference (or habit) level of consumption over time. [Jin and Zhou \(2008\)](#) and [He and Zhou \(2011, 2014\)](#) consider optimal portfolio choice under prospect theory. They focus on the impact of subjective probability weighting on optimal portfolio (not consumption) choice, developing an analytic solution method based on a quantile formulation. They do not consider endogenous updating of the reference level. Our model specification has the attractive feature that it allows to analyze both separately and jointly the effects on consumption and portfolio choice of loss aversion and of endogenous updating of the reference level, which are controlled in the model by separate parameters. Furthermore, our model nests traditional models, such as models with internal habit formation, with an exogenous minimum level of consumption, and with CRRA utility, as special (limiting) cases.

The remainder of this paper is structured as follows. Section 2 describes the economy. The agent’s instantaneous utility function is introduced in Section 3. Section 4 analytically derives the optimal consumption and portfolio choice. The properties of the optimal strategies are

explored in Section 5. Section 6 considers, as a robustness check, the optimal consumption and portfolio choice under a slightly alternative specification of the agent's instantaneous utility function. Finally, Section 7 concludes the paper. The proofs of the theorems and propositions and the details of the certainty equivalent computations are relegated to the Appendix.

2 The Economy

We define a continuous-time financial market following Karatzas and Shreve (1998) and Back (2010). Let $T > 0$ be a fixed finite terminal time. The uncertainty in the economy is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, on which is defined a standard N -dimensional Brownian motion $\{Z_t\}_{t \in [0, T]}$. Let the filtration $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t \in [0, T]}$ be the augmentation under \mathbb{P} of the natural filtration generated by the standard Brownian motion $\{Z_t\}_{t \in [0, T]}$. Throughout, (in)equalities between random variables are meant to hold \mathbb{P} -almost surely.

The financial market consists of an instantaneously risk-free asset and N risky stocks, which are traded continuously on the time horizon $[0, T]$. The price of the risk-free asset, B , evolves according to

$$\frac{dB_t}{B_t} = r_t dt, \quad B_0 = 1.$$

The scalar-valued risk-free rate process, r , is assumed to be \mathcal{F}_t -progressively measurable and uniformly bounded. The N -dimensional vector of risky stock prices, S , satisfies the following stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dZ_t, \quad S_0 = \mathbf{1}_N.$$

Here, $\mathbf{1}_N$ denotes an N -dimensional vector of all ones. The N -dimensional mean rate of return process, μ , and the $(N \times N)$ -matrix-valued volatility process, σ , are both assumed to be \mathcal{F}_t -progressively measurable and uniformly bounded.

We assume that, for some positive ϵ ,

$$\vartheta^\top \sigma_t \sigma_t^\top \vartheta \geq \epsilon \|\vartheta\|^2, \quad \text{for all } \vartheta \in \mathbb{R}^N. \quad (2.1)$$

Here, \top denotes the transpose sign. The strong non-degeneracy condition (2.1) implies that the inverse of σ_t exists and is bounded. The \mathcal{F}_t -progressively measurable market price of risk process, λ , solves the following equation:

$$\sigma_t \lambda_t \equiv \mu_t - r_t \mathbf{1}_N.$$

The unique positive-valued state price density process, M , can now be defined as follows:

$$M_t \equiv \exp \left\{ - \int_0^t r_s ds - \int_0^t \lambda_s^\top dZ_s - \frac{1}{2} \int_0^t \|\lambda_s\|^2 ds \right\}.$$

The economy is populated by a single price-taking agent endowed with initial wealth $W_0 \geq 0$. The agent's objective is to choose an \mathcal{F}_t -progressively measurable N -dimensional process π , referred to as the portfolio process and representing the dollar amounts invested in the N risky stocks, and an \mathcal{F}_t -progressively measurable process c , referred to as the consumption process, so as to maximize the expectation of lifetime utility.¹¹ We impose the following integrability conditions, which we assume throughout to be satisfied for any consumption and portfolio process:

$$\int_0^T \pi_t^\top \sigma_t \sigma_t^\top \pi_t dt < \infty, \quad \int_0^T |\pi_t (\mu_t - r_t \mathbf{1}_N)| dt < \infty, \quad \mathbb{E} \left[\int_0^T |c_t|^2 dt \right] < \infty.$$

The wealth process, W , associated with a consumption and portfolio strategy (c, π) satisfies the following *dynamic budget constraint*:

$$dW_t = \left(r_t W_t + \pi_t^\top \sigma_t \lambda_t - c_t \right) dt + \pi_t^\top \sigma_t dZ_t, \quad W_0 \geq 0 \text{ given.} \quad (2.2)$$

Equation (2.2) reveals that the agent's wealth equals initial wealth, plus trading gains, minus cumulative consumption. The total dollar amount invested in the risk-free asset at time $t \in [0, T]$ is given by $W_t - \pi_t^\top \mathbf{1}_N$. We call a consumption and portfolio strategy *admissible* if the associated wealth process is uniformly bounded from below. Then the *static budget constraint* is also satisfied; see, e.g., Karatzas and Shreve (1998, p. 91-92) for further details.

3 The Agent's Utility Function

This section introduces the agent's (instantaneous) utility function $u(c_t; \theta_t)$. Here, θ_t represents the agent's reference level to which consumption is compared. We assume that the agent derives utility from the difference between consumption c_t and the reference level θ_t . Specifically, following the prospect theory literature (see, e.g., Tversky and Kahneman, 1992), we assume that the agent's utility function $u(c_t; \theta_t)$ is represented by the two-part power utility function:

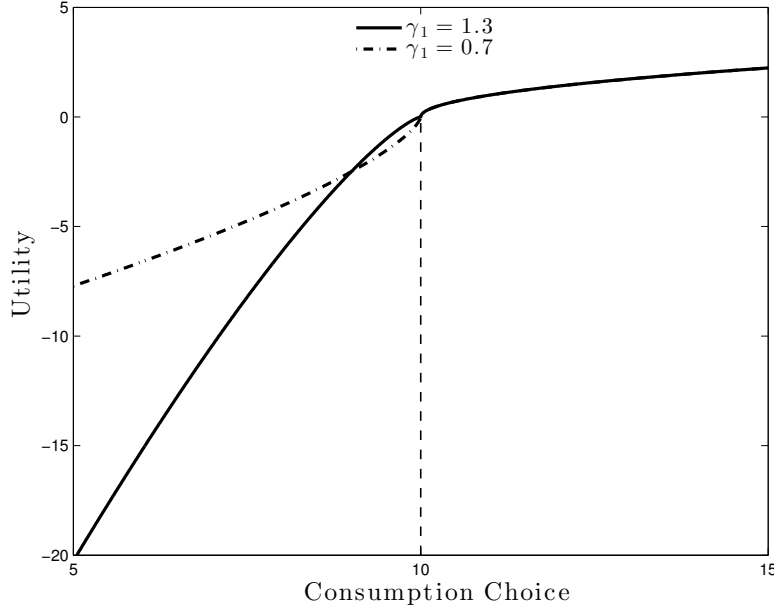
$$u(c_t; \theta_t) = v(c_t - \theta_t) \equiv \begin{cases} -\kappa (\theta_t - c_t)^{\gamma_1}, & \text{if } c_t < \theta_t; \\ (c_t - \theta_t)^{\gamma_2}, & \text{if } c_t \geq \theta_t. \end{cases} \quad (3.1)$$

Here, $\gamma_1 > 0$ and $\gamma_2 \in (0, 1)$ are curvature parameters, and $\kappa \geq 1$ stands for the loss aversion index. If consumption is larger (smaller) than the reference level, then the agent experiences a gain (loss).

Figure 1 illustrates the two-part power utility function (3.1) for $\gamma_1 = 1.3$ (solid line) and $\gamma_1 = 0.7$ (dash-dotted line). The figure shows that the two-part power utility function exhibits a kink at the reference level. The kink is due to the different treatment of gains and losses. We note that even in the case of $\kappa = 1$, the agent's utility function displays a kink at the reference level whenever $\gamma_1 \neq \gamma_2$.

¹¹The agent's utility function is introduced in the next section.

Figure 1.
Illustration of the two-part power utility function



Notes: The figure illustrates the two-part power utility function for $\gamma_1 = 1.3$ (solid line) and $\gamma_1 = 0.7$ (dash-dotted line). The agent's reference level is set equal to 10, the loss aversion index κ to 2.5 and γ_2 to 0.5.

A simple calculation shows that the two-part power utility function (3.1) is convex below the agent's reference level if $\gamma_1 \leq 1$, and concave otherwise. Convexity corresponds to risk-seeking behavior and concavity to risk-averse behavior.¹² Tversky and Kahneman (1992) found that the agent's utility function is convex in the loss domain. Table 1 reviews the empirical literature regarding the shape of the utility function for losses. The table shows that the literature is inconclusive as to whether the utility function is convex below the reference level. Among the mentioned studies, Etchart-Vincent (2004) explored the sensitivity of the agent's utility function to the magnitude of the underlying payoffs. She found that a larger proportion of the subjects exhibited concavity when facing large losses than when facing small losses. Etchart-Vincent (2004) argued that this finding may be due to the size of the losses at stake. Therefore, the current paper considers not only the case of a convex utility function in the loss domain ($0 < \gamma_1 \leq 1$), but also the case of a concave utility function in the loss domain ($\gamma_1 > 1$).

Inspired by the literature on internal habit formation (see, e.g., Constantinides, 1990; Detemple and Zapatero, 1992; Detemple and Karatzas, 2003), we assume that the agent's reference level evolves according to:

$$d\theta_t = (\beta c_t - \alpha \theta_t) dt, \quad \theta_0 \geq 0 \text{ given.}$$

¹²This statement is not true if probabilities are distorted (see Chateaufneuf and Cohen, 1994). For example, an S-shaped utility function and overweighting of small probabilities can together explain the fourfold pattern of risk attitudes: risk-averse behavior when gains have large probabilities and losses have small probabilities, and risk-seeking behavior when losses have large probabilities and gains have small probabilities.

Table 1.

Classification of the utility function for losses

Study	Shape of the utility function for losses			
	Convex	Concave	Linear	Mixed
Abdellaoui (2000)	42.5	20.0	25.0	12.5
Abdellaoui et al. (2005)	24.4	22.0	22.0	31.7
Abdellaoui, Bleichrodt, and Paraschiv (2007)	68.8	8.3	22.9	-
Booij and van de Kuilen (2009)	47.1	22.5	30.4	-
Etchart-Vincent (2004)*	37.1	25.7	25.7	11.4

*The reported results are for the case of large losses.

Notes: The table reviews the empirical literature regarding the shape of the utility function for losses. Numbers are expressed as a percentage of total subjects. All the mentioned studies use the trade-off method (see Wakker and Deneffe, 1996) to elicit the utility functions of the subjects.

Here, θ_0 denotes the agent's initial reference level, $\alpha \geq 0$ corresponds to the depreciation (or persistence) parameter, and $\beta \geq 0$ indexes the extent to which the current reference level responds to current consumption. The agent's reference level exhibits a low degree of depreciation (or a high degree of persistence) if α is low. The impact of current consumption on the current reference level increases as β increases. We can explicitly write the agent's reference level as follows:

$$\theta_s = \beta \int_t^s \exp\{-\alpha(s-u)\} c_u du + \exp\{-\alpha(s-t)\} \theta_t, \quad s \geq t \geq 0. \quad (3.2)$$

Equation (3.2) shows that the reference level can be decomposed into two components: a stochastic and a deterministic component. The parameter β measures the importance of the stochastic component relative to the deterministic component. In what follows, we refer to β as the endogeneity parameter. The stochastic component becomes more important as β increases. The first component on the right-hand side of equation (3.2) is an exponentially weighted integral of the agent's *own* past consumption choices (i.e., the reference level is backward-looking). We observe that the current reference level depends more on consumption in the recent past than on consumption in the distant past. The second component on the right-hand side of equation (3.2) is independent of past consumption choices and decreases exponentially at a rate of α .

The two-part utility function (3.1) is a member of the class of reference-dependent preferences introduced by Kőszegi and Rabin (2006, 2007). They assume that the agent's instantaneous utility function can be decomposed into two components. The first component represents classical utility from consumption; that is, utility derived from absolute levels of consumption. The second component captures reference-dependent gain-loss utility; that is, utility derived from the difference between classical consumption utility and the reference level of utility.

Specifically, [Kőszegi and Rabin \(2006, 2007\)](#) consider the following agent’s utility function:

$$u(c_t; \theta_t) = \eta \cdot m(c_t) + (1 - \eta) \cdot w(m(c_t) - m(\theta_t)). \quad (3.3)$$

Here, m stands for the classical consumption utility function, w denotes the gain-loss utility function and $\eta \in [0, 1]$ is a weight parameter controlling the relative importance of the two components. The two-part utility function (3.1) emerges as a special case of (3.3) if the gain-loss utility function w is represented by the two-part power utility function (3.1), the weight parameter η is set equal to zero and $m(c_t) = c_t$. Section 6 considers another special case of (3.3), where the weight parameter η is unequal to zero. [Kőszegi and Rabin \(2006, 2007\)](#) do not assume that the agent’s reference level is a weighted integral of past consumption choices. Instead, they assume that the agent’s reference level represents an expectation. Both [Kőszegi and Rabin \(2006, 2007\)](#) and our model assume that the reference level is chosen endogenously.¹³

The two-part power utility function (3.1) displays loss aversion in the sense that the disutility of a loss of one unit is κ times larger than the utility of a gain of one unit.¹⁴ There is, however, no agreed-upon definition of loss aversion in the literature. According to [Kahneman and Tversky \(1979\)](#), loss aversion refers to the fact that losses loom larger than same-sized gains, i.e., $-w(-x) > w(x)$ for all $x > 0$. A loss aversion index can then be defined as the mean or median value of $-w(-x)/w(x)$ over relevant x (see [Abdellaoui, Bleichrodt, and L’Haridon, 2008](#)). [Kőbberling and Wakker \(2005\)](#) define the loss aversion index as the ratio between the left-hand and right-hand derivative of the gain-loss utility function at the reference level. The loss aversion index κ is equal to the loss aversion index proposed by [Kőbberling and Wakker \(2005\)](#) if $\gamma_1 = \gamma_2$.

Finally, we note that the two-part power utility function (3.1) with reference level dynamics given by (3.2) includes several important special (limiting) cases. The internal habit formation model studied by [Constantinides \(1990\)](#) arises as a special case if the agent is infinitely loss averse. The assumption of infinite loss aversion implies that consumption is not allowed to fall below the reference level. If the reference level is also assumed to be exogenous, then the two-part power utility function reduces to a utility function with an exogenous minimum consumption level. Such a utility function has been studied by [Deelstra, Grasselli, and Koehl \(2003\)](#). The constant relative risk aversion (CRRA) utility function emerges as a special case if the reference level is equal to zero and consumption is non-negative. The CRRA utility function has been widely explored in the economics literature since at least [Merton \(1969\)](#).

4 The Consumption and Portfolio Choice Problem

This section derives the optimal consumption and portfolio choice. Section 4.1 formulates the agent’s maximization problem. To determine the optimal consumption and portfolio choice, we

¹³[Yogo \(2008\)](#) analyzes asset pricing implications of reference-dependent preferences, with an exogenously given reference level.

¹⁴As pointed out by [Wakker \(2010, p. 267\)](#), the degree of loss aversion depends on the monetary unit.

transform the agent's (primal) maximization problem into a dual problem. The technique that solves this dual problem is outlined in Section 4.2. Section 4.3 presents the optimal consumption choice and Section 4.4 gives the optimal portfolio choice.

4.1 The Agent's Maximization Problem

The agent's dynamic consumption and portfolio choice problem of Section 2 with the agent's utility function given in Section 3 can, by virtue of the martingale approach (Pliska, 1986; Karatzas, Lehoczky, and Shreve, 1987; Cox and Huang, 1989, 1991), be transformed into the following equivalent static variational problem:

$$\begin{aligned} & \underset{c}{\text{maximize}} && \mathbb{E} \left[\int_0^T \exp\{-\delta t\} v(c_t - \theta_t) dt \right] \\ & \text{subject to} && \mathbb{E} \left[\int_0^T M_t c_t dt \right] \leq W_0, \quad d\theta_t = (\beta c_t - \alpha \theta_t) dt, \\ & && c_t \geq \theta_t - L_t \quad \text{for all } t \in [0, T]. \end{aligned} \tag{4.1}$$

Here, $\delta \geq 0$ stands for the subjective rate of time preference. We require that consumption is not allowed to fall more than $L_t \geq 0$ below the agent's reference level θ_t .¹⁵ In addition, we assume that L_t only depends on time t (and not on the state of nature $\omega \in \Omega$).¹⁶ If $L_t = \exp\{-\alpha t\} \theta_0$, then consumption is guaranteed to be non-negative. We can view $\theta_t - L_t$ as the agent's *minimum consumption level*.

4.2 The Dual Technique

To derive the optimal consumption and portfolio choice in our model, we first apply the solution technique proposed by Schroder and Skiadas (2002). These authors show that a generic consumption and portfolio choice model with linear internal habit formation can be mechanically transformed into a *dual* consumption and portfolio choice model without linear internal habit formation.¹⁷ Hereinafter, we refer to the solution technique considered by Schroder and Skiadas (2002) as the dual technique. This section sketches the basic ideas underlying the dual technique. The Appendix provides more details.

The dual consumption and portfolio choice model [see problem (A1) in the Appendix] is solved in a dual financial market. This dual financial market is characterized by the dual state price density \widehat{M}_t , the dual (instantaneously) risk-free rate \widehat{r}_t , the dual volatility $\widehat{\sigma}_t$ and the dual

¹⁵In the case of risk-seeking behavior in the loss domain, the agent's maximization problem is ill-posed if consumption is not bounded from below (a maximization problem is called ill-posed if its supremum is infinite).

¹⁶One could argue that L_t should also depend on the agent's past consumption choices. However, this would complicate the agent's maximization problem considerably. We leave it for future research to explore the impact of an endogenous L_t on the agent's optimal consumption and portfolio choice.

¹⁷The consumption and portfolio choice model considered in the current paper indeed has a utility specification that incorporates linear internal habit formation.

market price of risk $\widehat{\lambda}_t$:

$$\begin{aligned}\widehat{M}_t &\equiv M_t(1 + \beta A_t), \\ \widehat{r}_t &\equiv \beta + \frac{r_t - \alpha\beta A_t}{1 + \beta A_t}, \\ \widehat{\sigma}_t &\equiv \sigma_t, \\ \widehat{\lambda}_t &\equiv \lambda_t - \frac{\beta}{1 + \beta A_t} \int_t^T \exp\{-(\alpha - \beta)(s - t)\} P_{t,s} \Psi_{t,s} ds.\end{aligned}$$

Here, $P_{t,s}$ corresponds to the time t price of a default-free unit discount bond that matures at time $s \geq t \geq 0$ and $\Psi_{t,s}$ stands for the time t volatility of the instantaneous return on such a bond (both in the primal financial market). We can view $A_t \geq 0$ as the time t price of a bond paying a continuous coupon:

$$A_t \equiv \frac{1}{M_t} \mathbb{E}_t \left[\int_t^T M_s \exp\{-(\alpha - \beta)(s - t)\} ds \right].$$

In case the investment opportunity set is constant, A_t only depends on time t . As a consequence, the optimal portfolio choice can be computed explicitly in this case (see Section 4.4).

Dual wealth \widehat{W}_t is subject to the following dynamic equation:

$$d\widehat{W}_t = \left(\widehat{r}_t \widehat{W}_t + \widehat{\pi}_t^\top \widehat{\sigma}_t \widehat{\lambda}_t - \widehat{c}_t \right) dt + \widehat{\pi}_t^\top \widehat{\sigma}_t dZ_t, \quad \widehat{W}_0 = \frac{W_0 - A_0 \theta_0}{1 + \beta A_0}. \quad (4.2)$$

Here, $\widehat{c}_t \equiv c_t - \theta_t$ stands for the agent's *surplus consumption choice* and $\widehat{\pi}_t$ denotes the dual portfolio choice. Dual wealth \widehat{W}_t is equal to the discounted value of future surplus consumption choices. Hence, we can view \widehat{W}_t as wealth needed to finance future gains and losses. In what follows, we refer to \widehat{W}_t as *surplus wealth*.

The condition of consumption being bounded from below in (4.1) implies that the agent's initial wealth W_0 must be sufficiently large to ensure the existence of an optimal consumption strategy. Specifically, we require

$$W_0 \geq -\mathbb{E} \left[\int_0^T \frac{\widehat{M}_t}{\widehat{M}_0} L_t dt \right] - \beta A_0 \mathbb{E} \left[\int_0^T \frac{\widehat{M}_t}{\widehat{M}_0} L_t dt \right] + A_0 \theta_0. \quad (4.3)$$

The right-hand side of equation (4.3) corresponds to initial wealth that is required to finance the minimum consumption stream $\{\theta_t - L_t\}_{t \in [0, T]}$. We note that W_0 is also required to be non-negative; see equation (2.2).

4.3 The Optimal Consumption Choice

This section derives the optimal consumption choice. We obtain the optimal consumption choice as follows. First, the agent's maximization problem (4.1) is converted into its dual problem (Section 4.2). The dual utility function is time-additive and separable. This fact facilitates the

derivation of the optimal consumption and portfolio choice. Second, the dual problem is solved using martingale techniques and by adapting to our setting with intertemporal consumption the solution technique as described by Basak and Shapiro (2001) and Berkelaar et al. (2004). The central idea of the latter solution technique is to split the agent's (dual, in our case) problem into two maximization problems: a gain part problem and a loss part problem. The optimal solution to each problem represents a *local* maximum of the dual problem. The *global* maximum of the dual problem is determined by comparing, in a particular way, the two local maxima. Finally, the optimal surplus consumption choice \widehat{c}_t^* is translated back into the agent's optimal consumption choice c_t^* . Theorem 1 below presents the optimal consumption choice c_t^* . We note that the theorem distinguishes between risk-averse and risk-seeking behavior in the loss domain. Indeed, in the case of risk-averse behavior in the loss domain, the utility function is concave below the reference level, whereas in the case of risk-seeking behavior in the loss domain, the utility function is convex in the loss domain.

Theorem 1. *Consider an agent with the two-part power utility function (3.1) and reference level dynamics (3.2) who solves the consumption and portfolio choice problem (4.1). Let θ^* be the agent's optimal reference level implied by substituting the (past) optimal consumption choice in (3.2) and let y be the Lagrange multiplier associated with the static budget constraint in (4.1). Define*

$$k_t \equiv \frac{y \exp\{\delta t\}}{\gamma_2} \quad \text{and} \quad l_t \equiv \frac{y \exp\{\delta t\}}{\kappa \gamma_1}.$$

Then:

- If the agent is risk-averse in the loss domain, the optimal consumption choice c_t^* at time $t \in [0, T]$ is given by

$$c_t^* = \begin{cases} \theta_t^* + \left(k_t \widehat{M}_t\right)^{\frac{1}{\gamma_2-1}}, & \text{if } \widehat{M}_t \leq \xi_t; \\ \theta_t^* - \left[\left(l_t \widehat{M}_t\right)^{\frac{1}{\gamma_1-1}} \wedge L_t\right], & \text{if } \widehat{M}_t > \xi_t. \end{cases}$$

The threshold ξ_t is determined in such a way that $f(\xi_t) = 0$ where the function f is defined as follows:

$$f(x) \equiv \exp\{-\delta t\} (1 - \gamma_2) (k_t x)^{\frac{\gamma_2}{\gamma_2-1}} + \kappa \exp\{-\delta t\} \left[\left(l_t x\right)^{\frac{1}{\gamma_1-1}} \wedge L_t\right]^{\gamma_1} - yx \left[\left(l_t x\right)^{\frac{1}{\gamma_1-1}} \wedge L_t\right]. \quad (4.4)$$

- If the agent is risk-seeking in the loss domain, the optimal consumption choice c_t^* at time $t \in [0, T]$ is given by

$$c_t^* = \begin{cases} \theta_t^* + \left(k_t \widehat{M}_t\right)^{\frac{1}{\gamma_2-1}}, & \text{if } \widehat{M}_t \leq \xi_t; \\ \theta_t^* - L_t, & \text{if } \widehat{M}_t > \xi_t. \end{cases}$$

The threshold ξ_t is determined in such a way that $g(\xi_t) = 0$ where the function g is defined as follows:

$$g(x) \equiv \exp\{-\delta t\} (1 - \gamma_2) (k_t x)^{\frac{\gamma_2}{\gamma_2 - 1}} + \kappa \exp\{-\delta t\} L_t^{\gamma_1} - y x L_t. \quad (4.5)$$

The Lagrange multiplier y is chosen such that the static budget constraint holds with equality.

Theorem 1 shows that the agent optimally chooses to divide the states of the economy into two categories: insured states (good to intermediate economic scenarios or, equivalently, low to intermediate state prices) and uninsured states (bad economic scenarios or high state prices). In insured states, consumption is guaranteed to be larger than the reference level, while in uninsured states, consumption is smaller than the reference level. The optimal consumption choice is, however, never equal to the reference level. Section 5 further explores the properties of the optimal consumption choice.

4.3.1 Comparative Statics

The threshold ξ_t and the Lagrange multiplier y depend on the underlying agent's preference parameters. Proposition 1 summarizes the impact of an increase in the agent's preference parameters on the threshold ξ_t and the Lagrange multiplier y , ceteris paribus.

Proposition 1. *Consider an agent with the two-part power utility function (3.1) and reference level dynamics (3.2) who solves the consumption and portfolio choice problem (4.1). Then:*

- *All else being equal, if the loss aversion index κ increases, then both the threshold ξ_t and the Lagrange multiplier y increase.*
- *All else being equal, if the agent's initial reference level θ_0 increases, then the threshold ξ_t decreases and the Lagrange multiplier y increases.*

Suppose that initial surplus wealth \widehat{W}_0 is non-negative.

- *All else being equal, if the depreciation parameter α increases, then the threshold ξ_t increases and the Lagrange multiplier y decreases.*
- *All else being equal, if the endogeneity parameter β increases, then the threshold ξ_t decreases and the Lagrange multiplier y increases.*

Proposition 1 shows that when the agent becomes more afraid of incurring losses, the probability of consumption falling below the reference level decreases. At the same time, the agent must give up some upward potential to finance the new consumption profile. When the agent's initial reference level increases (or the depreciation parameter α decreases or the endogeneity parameter β increases), more wealth is required to finance future reference levels. As a consequence, the probability of incurring a loss increases.

4.4 The Optimal Portfolio Choice

To derive the optimal portfolio choice, we first need to derive the agent's optimal wealth W_t^* . As pointed out in the Appendix (see Proposition 4), the agent's optimal wealth W_t^* can be decomposed as follows:

$$W_t^* = \widehat{W}_t^* + \widetilde{W}_t^*. \quad (4.6)$$

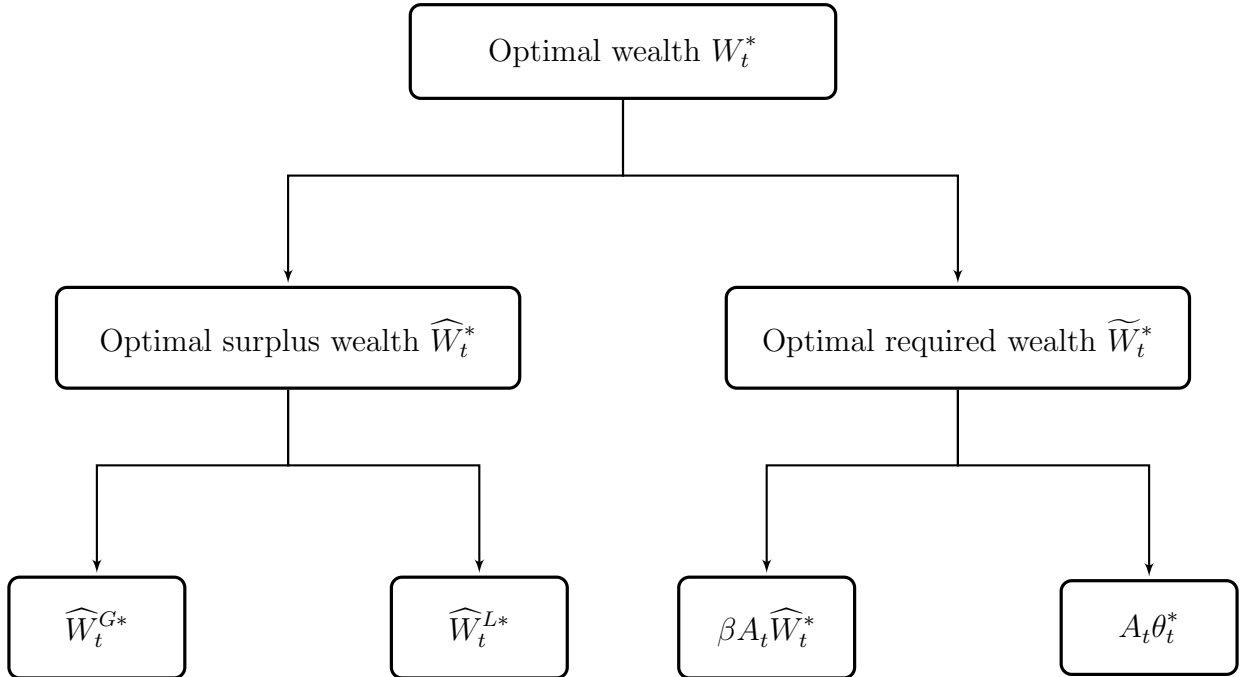
Here, \widehat{W}_t^* denotes optimal surplus wealth, and \widetilde{W}_t^* stands for wealth required to finance future optimal reference levels. We refer to \widetilde{W}_t^* as optimal *required wealth*. Optimal surplus wealth \widehat{W}_t^* and optimal required wealth \widetilde{W}_t^* can be further decomposed as follows:

$$\widehat{W}_t^* = \widehat{W}_t^{G*} + \widehat{W}_t^{L*} \quad \text{and} \quad \widetilde{W}_t^* = \beta A_t \widehat{W}_t^* + A_t \theta_t^*. \quad (4.7)$$

Here, \widehat{W}_t^{G*} denotes wealth required to finance future optimal gains, \widehat{W}_t^{L*} corresponds to wealth required to finance future optimal losses, $\beta A_t \widehat{W}_t^*$ stands for wealth required to finance the stochastic part of future optimal reference levels, and $A_t \theta_t^*$ represents wealth required to finance the deterministic part of future optimal reference levels. Figure 2 illustrates the decomposition of the agent's optimal wealth W_t^* .

Figure 2.

Decomposition of the agent's optimal wealth W_t^*



Notes: The figure illustrates the decomposition of the agent's optimal wealth W_t^* .

Proposition 2 below presents \widehat{W}_t^{G*} and \widehat{W}_t^{L*} for the case of a constant investment opportunity set (i.e., $r_t = r$, $\sigma_t = \sigma$ and $\lambda_t = \lambda$). The general expressions for \widehat{W}_t^{G*} and \widehat{W}_t^{L*} are given in

the Appendix.

Proposition 2. Consider an agent with the two-part power utility function (3.1) and reference level dynamics (3.2) who solves the consumption and portfolio choice problem (4.1) assuming a constant investment opportunity set. Let \mathcal{N} denote the cumulative distribution function of a standard normal random variable. Define Γ_u , Π_u , $d_1(x)$, $d_2(x)$ and $d_3(x)$ as follows:

$$\Gamma_u = \frac{\delta - \gamma_2 \widehat{r}_u}{1 - \gamma_2} - \frac{1}{2} \frac{\gamma_2}{(1 - \gamma_2)^2} \|\lambda\|^2, \quad \Pi_u = \frac{\delta - \gamma_1 \widehat{r}_u}{1 - \gamma_1} - \frac{1}{2} \frac{\gamma_1}{(1 - \gamma_1)^2} \|\lambda\|^2,$$

$$d_1(x) = \frac{1}{\|\lambda\| \sqrt{s-t}} \left[\log(x) - \log(\widehat{M}_t) + \int_t^s \widehat{r}_u du - \frac{1}{2} \|\lambda\|^2 (s-t) \right],$$

$$d_2(x) = d_1(x) + \frac{\|\lambda\|}{1 - \gamma_2} \sqrt{s-t}, \quad d_3(x) = d_1(x) + \frac{\|\lambda\|}{1 - \gamma_1} \sqrt{s-t}.$$

Then:

- If the agent is risk-averse in the loss domain, we find

$$\widehat{W}_t^{G*} = \left(k_t \widehat{M}_t \right)^{\frac{1}{\gamma_2 - 1}} \int_t^T \exp \left\{ - \int_t^s \Gamma_u du \right\} \mathcal{N} [d_2(\xi_s)] ds,$$

$$\begin{aligned} \widehat{W}_t^{L*} &= \left(l_t \widehat{M}_t \right)^{\frac{1}{\gamma_1 - 1}} \int_t^T \exp \left\{ - \int_t^s \Pi_u du \right\} \left(\mathcal{N} [d_3(\zeta_s \vee \xi_s)] - \mathcal{N} [d_3(\xi_s)] \right) ds \\ &\quad - \int_t^T \exp \left\{ - \int_t^s \widehat{r}_u du \right\} L_s \mathcal{N} [-d_1(\zeta_s \vee \xi_s)] ds. \end{aligned}$$

Here, $\zeta_s \equiv \exp \{ \delta s \} \gamma_1 \kappa L_s^{\gamma_1 - 1} y^{-1}$. The threshold ξ_s is determined in such a way that $f(\xi_s) = 0$ where the function f is given by equation (4.4).

- If the agent is risk-seeking in the loss domain, we find

$$\widehat{W}_t^{G*} = \left(l_t \widehat{M}_t \right)^{\frac{1}{\gamma_2 - 1}} \int_t^T \exp \left\{ - \int_t^s \Gamma_u du \right\} \mathcal{N} [d_2(\xi_s)] ds,$$

$$\widehat{W}_t^{L*} = \int_t^T \exp \left\{ - \int_t^s \widehat{r}_u du \right\} L_s \mathcal{N} [-d_1(\xi_s)] ds.$$

The threshold ξ_s is determined in such a way that $g(\xi_s) = 0$ where the function g is given by equation (4.5).

When the dual state price density tends to zero (so that the probability of the dual state price density \widehat{M}_s being smaller than the threshold ξ_s approaches one), optimal surplus wealth \widehat{W}_t^* converges to the optimal wealth of an agent with CRRA utility. Hence, in good economic scenarios, the agent behaves like a CRRA agent.

The optimal dual portfolio choice can be constructed using classical hedging arguments. We explicitly determine the optimal dual portfolio choice for the case of a constant investment opportunity set. To this end, it is convenient to express \widehat{W}_t^* as a function of time t and the dual

state price density \widehat{M}_t ; that is, $\widehat{W}_t^* = h(t, \widehat{M}_t)$ for some (regular) function h . Straightforward application of Itô's Lemma to the function h yields

$$d\widehat{W}_t^* = \left[\frac{\partial h}{\partial t} - \frac{\partial h}{\partial \widehat{M}_t} \widehat{M}_t \widehat{r}_t + \frac{1}{2} \frac{\partial^2 h}{\partial \widehat{M}_t^2} \widehat{M}_t^2 \|\lambda\|^2 \right] dt - \frac{\partial h}{\partial \widehat{M}_t} \widehat{M}_t \widehat{\lambda}^\top dZ_t. \quad (4.8)$$

Comparing the diffusion part of the dynamic budget constraint (4.2) with the diffusion part of equation (4.8) yields the dual optimal portfolio choice:

$$\widehat{\pi}_t^* = -\frac{\partial h}{\partial \widehat{M}_t} \widehat{M}_t \widehat{\lambda}^\top \widehat{\sigma}^{-1}. \quad (4.9)$$

The agent's optimal (primal) portfolio choice follows from [Schroder and Skiadas \(2002\)](#):

$$\pi_t^* = \widehat{\pi}_t^* + \beta A_t \widehat{\pi}_t^*. \quad (4.10)$$

The optimal dual portfolio choice $\widehat{\pi}_t^*$ can be further decomposed as follows:

$$\widehat{\pi}_t^* = \widehat{\pi}_t^{G*} + \widehat{\pi}_t^{L*}.$$

Here, $\widehat{\pi}_t^{G*}$ denotes the optimal dual portfolio choice that finances gains, and $\widehat{\pi}_t^{L*}$ corresponds to the optimal dual portfolio choice that finances losses. [Theorem 2](#) below presents $\widehat{\pi}_t^{G*}$ and $\widehat{\pi}_t^{L*}$ for the case of a constant investment opportunity set. This theorem follows from application of equation (4.9). The optimal primal portfolio choice then follows from equation (4.10).

Theorem 2. *Consider an agent with the two-part power utility function (3.1) and reference level dynamics (3.2) who solves the consumption and portfolio choice problem (4.1) assuming a constant investment opportunity set. Let ϕ denote the standard normal probability density function. Then:*

- *If the agent is risk-averse in the loss domain, we find*

$$\begin{aligned} \widehat{\pi}_t^{G*} &= \widehat{\lambda}^\top \widehat{\sigma}^{-1} \left[\frac{1}{1 - \gamma_2} \widehat{W}_t^{G*} + \left(k_t \widehat{M}_t \right)^{\frac{1}{\gamma_2 - 1}} \int_t^T \exp \left\{ - \int_t^s \Gamma_u du \right\} \frac{\phi [d_2(\xi_s)]}{\|\lambda\| \sqrt{s - t}} ds \right], \\ \widehat{\pi}_t^{L*} &= \widehat{\lambda}^\top \widehat{\sigma}^{-1} \left[\frac{1}{\gamma_1 - 1} \left(\widehat{W}_t^{L*} + \int_t^T \exp \left\{ - \int_t^s \widehat{r}_u du \right\} L_s \mathcal{N} [-d_1(\zeta_s \vee \xi_s)] ds \right) \right. \\ &\quad + \left(l_t \widehat{M}_t \right)^{\frac{1}{\gamma_1 - 1}} \int_t^T \exp \left\{ - \int_t^s \Pi_u du \right\} \frac{\phi [d_2(\zeta_s \vee \xi_s)] - \phi [d_2(\xi_s)]}{\|\lambda\| \sqrt{s - t}} ds \\ &\quad \left. + \int_t^T \exp \left\{ - \int_t^s \widehat{r}_u du \right\} L_s \frac{\phi [-d_1(\zeta_s \vee \xi_s)]}{\|\lambda\| \sqrt{s - t}} ds \right]. \end{aligned}$$

- *If the agent is risk-seeking in the loss domain, we find*

$$\widehat{\pi}_t^{G*} = \widehat{\lambda}^\top \widehat{\sigma}^{-1} \left[\frac{1}{1 - \gamma_2} \widehat{W}_t^{G*} + \left(k_t \widehat{M}_t \right)^{\frac{1}{\gamma_2 - 1}} \int_t^T \exp \left\{ - \int_t^s \Gamma_u du \right\} \frac{\phi [d_2(\xi_s)]}{\|\lambda\| \sqrt{s - t}} ds \right],$$

$$\widehat{\pi}_t^{L*} = \widehat{\lambda}^\top \widehat{\sigma}^{-1} \int_t^T \exp \left\{ - \int_t^s \widehat{r}_u du \right\} L_s \frac{\phi[-d_1(\xi_s)]}{\|\lambda\| \sqrt{s-t}} ds.$$

Theorem 2 reveals that in good economic scenarios, the optimal dual portfolio strategy $\widehat{\pi}_t^*$ can be approximated by $\widehat{\lambda}^\top \widehat{\sigma}^{-1} / (1 - \gamma_2) \widehat{W}_t^*$. In these economic scenarios, the agent behaves like a CRRA agent and invests a *constant* proportion of surplus wealth in risk-bearing assets.

5 Analysis of the Solution

With the analytical solutions and comparative statics to the general consumption and portfolio choice problem provided in Section 4 (and the Appendix), we proceed in this section to their numerical analysis. Section 5.1 introduces the underlying assumptions and discusses the key parameter values used in the numerical analysis. Section 5.2 illustrates the agent's optimal consumption and portfolio choice. Finally, Section 5.3 conducts a welfare analysis.

5.1 Assumptions and Key Parameter Values

We allow the agent to invest his wealth in a risk-free asset and a single risky stock. The investment opportunity set is assumed to be constant (i.e., $r_t = r$, $\sigma_t = \sigma$ and $\lambda_t = \lambda$). The equity premium $\sigma\lambda = \mu - r$ is set at 4%. The risk-free rate r is set at 1%, and the volatility of innovations to the risky stock price σ is set at 20%. These estimates coincide with the estimates reported by Gomes et al. (2008).

The terminal time T is set equal to 20 years. We interpret T as the total number of years of retirement. The agent's initial wealth W_0 can then be viewed as total pension (or financial) wealth at the age of retirement.¹⁸ For the ease of illustration, we assume that the agent retires at the age of 65.

The loss aversion index κ is set equal to 2.5. The estimates of the median loss aversion index reported in the literature vary from 1 to 5 (see, e.g., Abdellaoui et al., 2008). The degree of loss aversion largely differs among individuals, and typically depends on the model. In the welfare analysis, we consider, among other things, the impact of a change in the loss aversion index κ on the agent's welfare. Finally, the subjective rate of time preference δ is set equal to 1%.

5.2 The Optimal Consumption and Portfolio Choice

5.2.1 Loss Aversion Only

This section illustrates the optimal consumption and portfolio choice of a loss averse agent without endogenous updating of the agent's reference level (i.e., the endogeneity parameter β is set equal to zero). In addition, we assume that the agent's reference level is constant (i.e.,

¹⁸In the numerical analysis, we set W_0 equal to 500 ($\times 1,000$ dollars) units, and we report our results *relative* to W_0 .

the depreciation parameter α is also set equal to zero). Inspired by Barberis, Huang, and Santos (2001), the agent's constant reference level $\theta_t = \theta$ is assumed to be equal to the level of consumption that would be obtained if the agent's initial wealth W_0 was kept in the money market account for the entire retirement phase.¹⁹ The assumption here is that the agent is likely to be disappointed if consumption is less than the payment he would receive from a fixed annuity. The agent's constant reference level θ solves the following equation:

$$W_0 = \theta \int_0^T \exp\{-rt\} dt \equiv \theta A_0. \quad (5.1)$$

Simple algebra yields $\theta = 5.5\% \cdot W_0$; that is, the annuity factor $A_0 \equiv \int_0^T \exp\{-rt\} dt$ is equal to $1/5.5\% \approx 18 < T = 20$. Equation (5.1) implies that initial surplus wealth $\widehat{W}_0 \equiv \widehat{W}_0^G - \widehat{W}_0^L$ is equal to zero. This assertion follows from equations (4.6) and (4.7) with $\alpha = \beta = 0\%$. Put differently, initial wealth required to finance future gains \widehat{W}_0^G is equal to initial wealth required to finance future losses \widehat{W}_0^L . We note that \widehat{W}_0^G and \widehat{W}_0^L are not equal to zero unless the agent is infinitely loss averse.

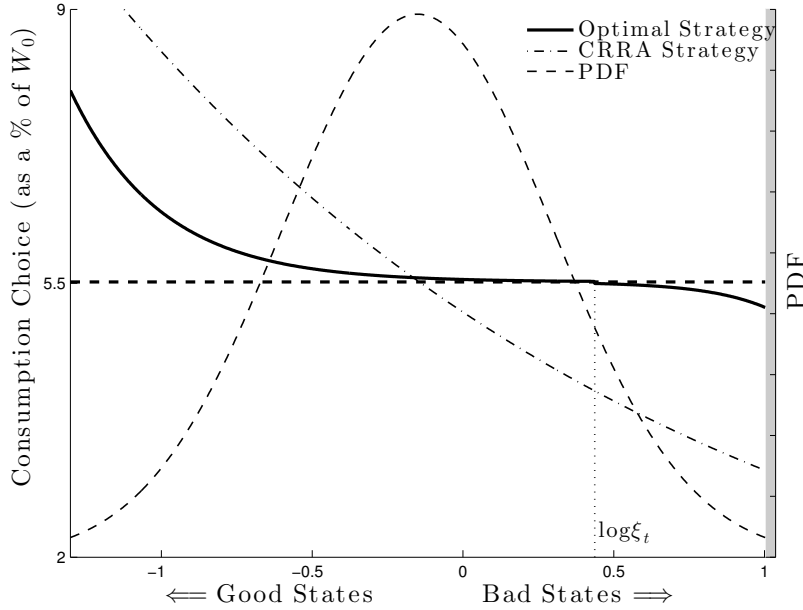
Figure 3 illustrates the optimal consumption choice (expressed as a percentage of the agent's initial wealth W_0) at age 70 (i.e., $t = 5$) as a function of the then-current log state price density for the case of risk-averse behavior in the loss domain. Here, consumption is constrained to be non-negative; that is, $L = \theta$. Under the optimal choice, the loss averse agent seeks protection against consumption losses due to financial shocks, thus inducing a (soft) guarantee on consumption. The agent optimally desires to maintain consumption above the reference level, but under really adverse circumstances this (soft) guarantee on consumption cannot be maintained. As a direct consequence, we can divide the states of the economy into two categories: good to intermediate states (i.e., $\log M_t \leq \log \xi_t$) and bad states (i.e., $\log M_t > \log \xi_t$). In good to intermediate states, optimal consumption is guaranteed to be larger than the reference level, while in bad states, optimal consumption is smaller than the reference level. The dotted line shows the probability density function (PDF) of the then-current log state price density conditional upon information available at the age of retirement. The probability of consumption being smaller than the reference level can be controlled by choosing appropriate values for the preference parameters. We observe that the optimal consumption profile displays a 90° rotated S-shaped pattern with a discontinuity at the point $\log M_t = \log \xi_t$. Hence, optimal consumption is never equal to the reference level. The dash-dotted line illustrates the consumption choice of an agent with CRRA utility. The relative risk aversion coefficient γ is set equal to two. The (log) consumption choice of a CRRA agent varies linearly with the (log) state price density. As a consequence, for typical values of the relative risk aversion coefficient γ , a CRRA agent incurs more frequently a loss than a loss averse agent (where we define gains and losses relative to the reference level).

Next, Figure 4 displays the optimal consumption choice (expressed as a percentage of the

¹⁹Barberis et al. (2001) argue that the risk-free interest rate serves as a natural benchmark for evaluating gains and losses. In our context, this assumption implies that the agent is likely to be disappointed if consumption is less than the payment he would receive from a fixed annuity.

Figure 3.

Consumption profile for the case of risk-averse behavior in the loss domain



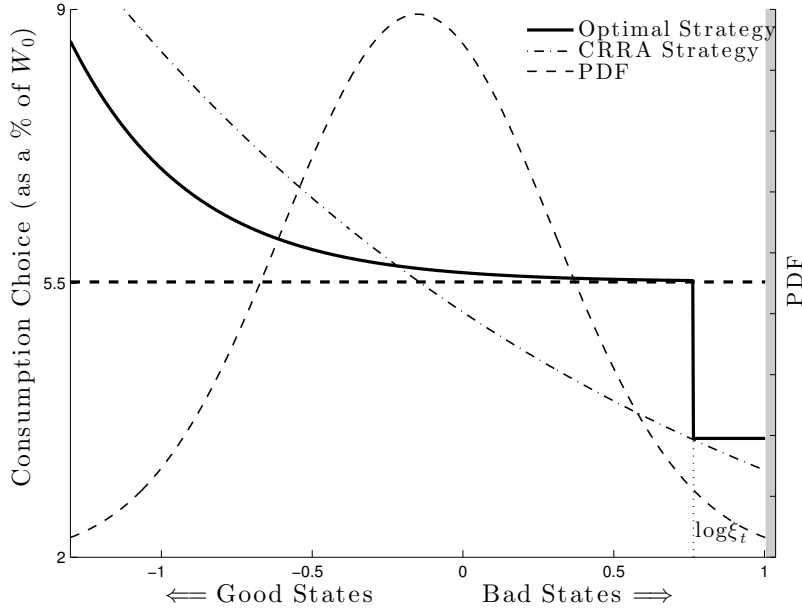
Notes: The figure shows the optimal consumption choice (expressed as a percentage of the agent’s initial wealth W_0) at age 70 as a function of the then-current log state price density. The curvature parameter γ_1 (γ_2) is set equal to 1.2 (0.7). Consumption is constrained to be non-negative by taking $L = \theta$. The dashed line corresponds to the agent’s reference level (expressed as a percentage of W_0). The dotted line shows the probability density function (PDF) of the then-current log state price density conditional upon information available at the age of retirement. The dash-dotted line illustrates the consumption choice (expressed as a percentage of W_0) of an agent with CRRA utility. The relative risk aversion coefficient γ is set equal to two.

agent’s initial wealth W_0) at age 70 as a function of the then-current log state price density for the case of risk-seeking behavior in the loss domain. Here, consumption is allowed to fall 2% point below the (normalized) reference level θ/W_0 . We observe again that, because of loss aversion, the agent has a strong preference to maintain consumption above the reference level. As in the case of risk-averse behavior in the loss domain, we can divide the states of the economy into two categories: good to intermediate states (i.e., $\log M_t \leq \log \xi_t$) and bad states (i.e., $\log M_t > \log \xi_t$). In good to intermediate states, optimal consumption is guaranteed to be larger than the reference level, while in bad states, optimal consumption is equal to the minimum consumption level $\theta - L$. We also observe that at the threshold $\log M_t = \log \xi_t$, optimal consumption jumps to the lower bound $\theta - L$. This behavior can be explained by the fact that the agent is risk-seeking in the loss domain.

Figure 5 shows the optimal portfolio choice (i.e., the total dollar amount invested in the risky stock) at age 70 as a function of the then-current log state price density for the case of risk-averse behavior in the loss domain. The optimal portfolio choice is expressed as a percentage of the agent’s initial wealth W_0 . We observe that the optimal portfolio profile displays a U-shaped pattern: the total dollar amount invested in the risky stock will be lower

Figure 4.

Consumption profile for the case of risk-seeking behavior in the loss domain



Notes: The figure shows the optimal consumption choice (expressed as a percentage of the agent's initial wealth W_0) at age 70 as a function of the then-current log state price density. The curvature parameter γ_1 (γ_2) is set equal to 0.8 (0.6). Consumption is allowed to fall 2% point below the (normalized) reference level θ/W_0 . The dashed line corresponds to the agent's reference level (expressed as a percentage of W_0). The dotted line shows the probability density function (PDF) of the then-current log state price density conditional upon information available at the age of retirement. The dash-dotted line illustrates the consumption choice (expressed as a percentage of W_0) of an agent with CRRA utility. The relative risk aversion coefficient γ is set equal to two.

in intermediate economic scenarios than in good or bad economic scenarios. When the (non-log) state price density tends to zero, the fraction of surplus wealth \widehat{W}_t^* invested in the risky stock converges to the constant $\lambda/[\sigma(1-\gamma_2)]$. Hence, in good economic scenarios, the optimal portfolio choice behaves in a similar fashion as the portfolio choice of a CRRA agent.²⁰ We note that $W_t^* - \widehat{W}_t^* = A_t\theta$ is fully invested in the money market account. When the state price density is relatively high, the fraction of surplus wealth invested in the risky stock can be approximated by the constant $\lambda/[\sigma(1-\gamma_1)] < 0$.²¹ Not only in good but also in bad economic scenarios, a loss averse agent behaves like a CRRA agent. In intermediate economic scenarios, the total dollar amount invested in the risky stock is relatively small.

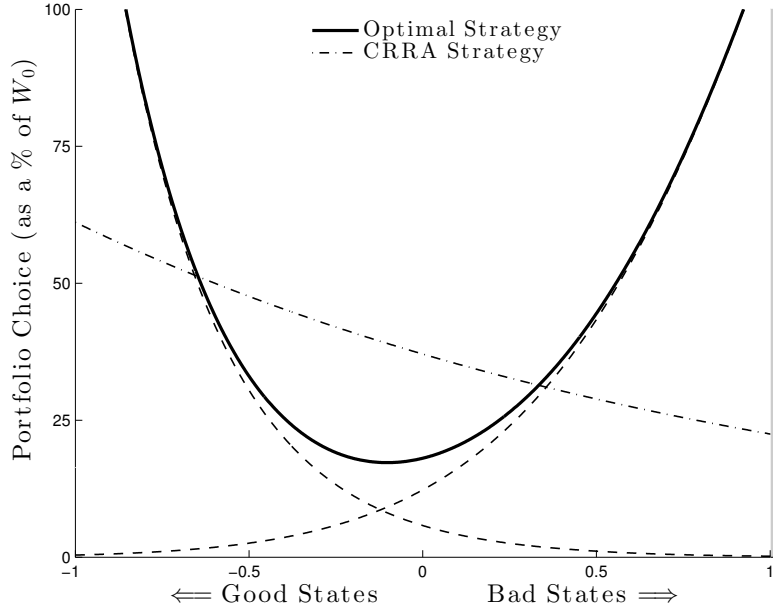
Figure 6 shows the optimal portfolio choice (i.e., the total dollar amount invested in the risky stock) at age 70 as a function of the then-current log state price density for the case of risk-seeking behavior in the loss domain. The portfolio choice is expressed as a percentage of

²⁰This is not directly visible in Figure 5, where the portfolio choice of the CRRA agent does not match the portfolio choice of the loss averse agent in good (or bad) states, because the relative risk aversion coefficient γ (CRRA agent) differs from its counterpart $1 - \gamma_i$ (loss averse agent) specified by the curvature parameters γ_i , $i = 1, 2$, and because total wealth differs from surplus wealth.

²¹We note that in bad states (i.e., high state prices), surplus wealth is negative.

Figure 5.

Portfolio profile for the case of risk-averse behavior in the loss domain



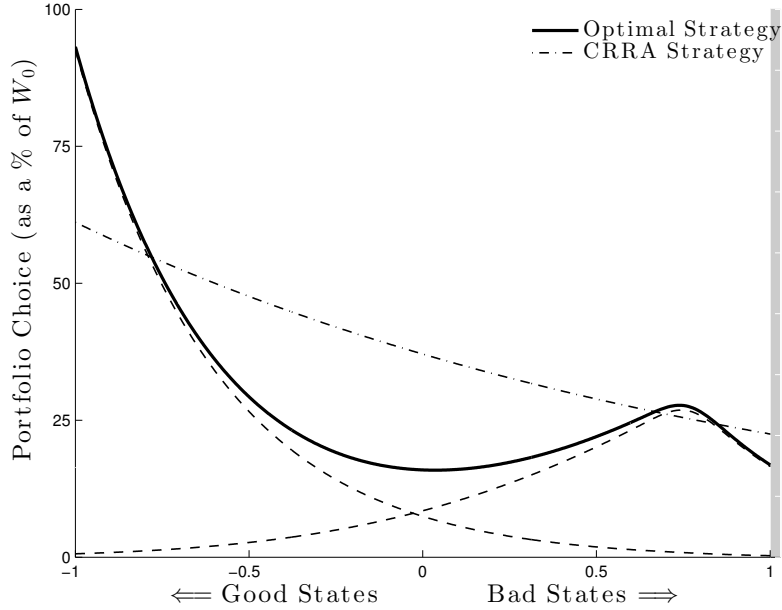
Notes: The figure shows the optimal portfolio choice (i.e., the total dollar amount invested in the risky stock) at age 70 as a function of the then-current log state price density. The portfolio choice is expressed as a percentage of the agent’s initial wealth W_0 . The curvature parameter γ_1 (γ_2) is set equal to 1.2 (0.7). Consumption is constrained to be non-negative; that is, $L = \theta$. The dash-dotted line illustrates the portfolio choice (expressed as a percentage of W_0) of an agent with CRRA utility. The relative risk aversion coefficient γ is set equal to two. The increasing dotted line represents $\hat{\pi}_t^{L^*}/W_0$ while the decreasing dotted line corresponds to $\hat{\pi}_t^{G^*}/W_0$.

the agent’s initial wealth W_0 . As in the case of risk-averse behavior in the loss domain, the optimal portfolio profile displays (primarily) a U-shaped pattern. When the state price density tends to zero, the fraction of surplus wealth invested in the risky stock converges to the constant $\lambda/[\sigma(1-\gamma_2)]$. Hence, in good economic scenarios, the portfolio choice of a loss averse agent behaves in a similar fashion as the portfolio choice of a CRRA agent. When the state price density tends to infinity, the fraction of surplus wealth invested in the risky stock ultimately converges to zero. Indeed, in bad economic scenarios, the minimum consumption level $\theta - L$ must be guaranteed.

Figure 7 shows the optimal portfolio choice measured as a fraction of *total wealth* invested in the risky stock at age 70 as a function of the then-current log state price density. We recall that Figures 5 and 6 show the optimal portfolio choice measured as a fraction of *initial wealth* invested in the risky stock. We observe that the optimal portfolio profile still displays (primarily) a U-shaped pattern. The portfolio choice of a CRRA agent is no longer a decreasing line but a straight line: a CRRA agent always invests a constant fraction $\lambda/(\sigma\gamma)$ of total wealth in the risky stock, irrespective of the state of the economy.

Figure 6.

Portfolio profile for the case of risk-seeking behavior in the loss domain



Notes: The figure shows the optimal portfolio choice (i.e., the total dollar amount invested in the risky stock) at age 70 as a function of the then-current log state price density. The portfolio choice is expressed as a percentage of the agent’s initial wealth W_0 . The curvature parameter γ_1 (γ_2) is set equal to 0.8 (0.6). Consumption is allowed to fall 2% point below the (normalized) reference level θ/W_0 . The dash-dotted line illustrates the portfolio choice (expressed as a percentage of W_0) of an agent with CRRA utility. The relative risk aversion coefficient γ is set equal to two. The increasing dotted line represents $\hat{\pi}_t^{L^*}/W_0$ while the decreasing dotted line corresponds to $\hat{\pi}_t^{G^*}/W_0$.

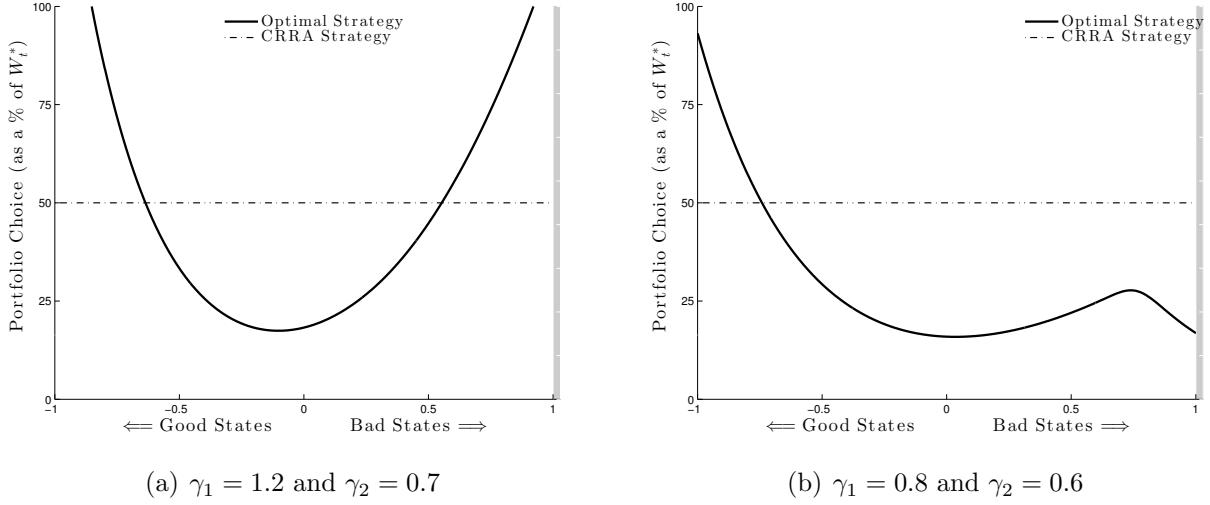
5.2.2 Loss Aversion and Endogenous Updating

This section considers the case where the loss averse agent endogenously updates his reference level over time. We assume that the endogeneity parameter β as well as the depreciation parameter α are equal to 20%. Furthermore, we assume that the initial reference level θ_0 is equal to 5.5% of the agent’s initial wealth W_0 , and L_t is equal to the initial reference level (i.e., $L_t = L = \theta_0$). These parameter values imply that initial surplus wealth \widehat{W}_0 is equal to zero.

Figure 8 illustrates the impact of a *positive* shock in initial wealth on median consumption for the case of risk-averse behavior in the loss domain. The left panel applies to the case in which the loss averse agent endogenously updates his reference level over time (i.e., $\alpha = \beta = 20\%$), while the right panel displays the case of no endogenous updating (i.e., $\alpha = \beta = 0\%$). The dash-dotted lines in both panels represent the agent’s median consumption choice *with* the shock in initial wealth. We observe that with endogenous updating a financial shock is gradually absorbed into future consumption (i.e., consumption adjusts sluggishly to financial shocks): the impact of a financial shock on consumption is smoothed over time, having a larger impact in the distant future than in the near future. By contrast, in the case of no endogenous updating, a financial shock is directly absorbed into future consumption, leading to an even distribution

Figure 7.

Fraction of total wealth invested in the risky stock



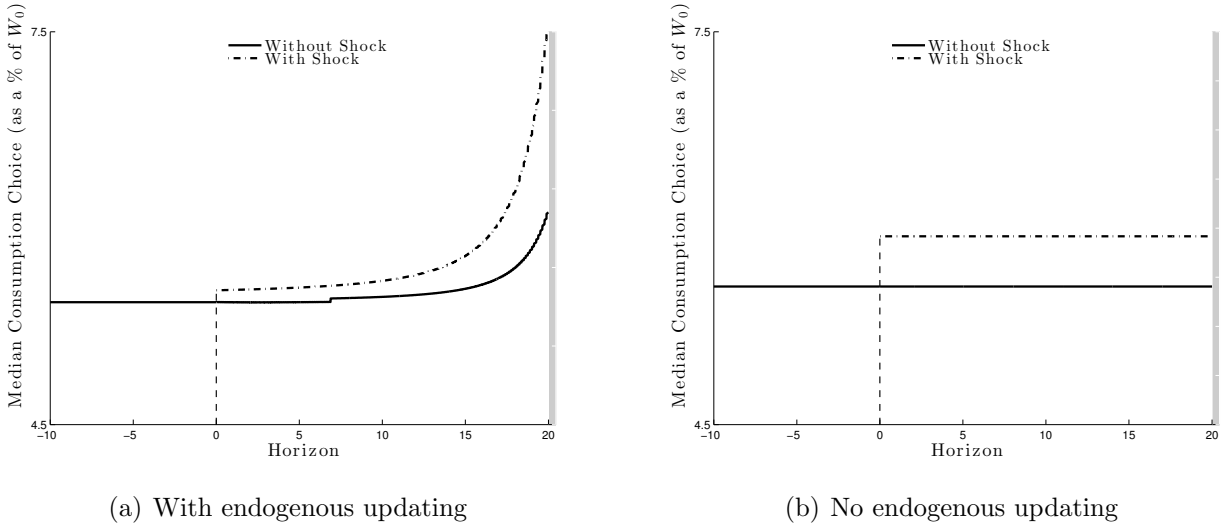
Notes: The figure shows the optimal portfolio choice measured as a fraction of total wealth invested in the risky stock at age 70 as a function of the then-current log state price density. Panel (a) displays the case of risk-averse behavior in the loss domain (taking, as before, $\gamma_1 = 1.2$ and $\gamma_2 = 0.7$), while panel (b) displays the case of risk-seeking behavior in the loss domain (taking, as before, $\gamma_1 = 0.8$ and $\gamma_2 = 0.6$). The dash-dotted line illustrates the portfolio choice of a CRRA agent. The relative risk aversion coefficient γ is set equal to two.

of the shock's impact on future consumption choice.

Figure 9 shows the optimal consumption choice (expressed as a percentage of the agent's initial wealth W_0) at age 70 as a function of the then-current log state price density and the then-current reference level. Indeed, we note that the optimal consumption profile depends not only the then-current state price density but also on the then-current reference level (i.e., the optimal consumption profile is path-dependent). The threshold ξ_t is however state independent. The agent is assumed to be risk-averse in the loss domain. The figure shows that the optimal consumption choice increases with the reference level, and decreases with the state price density. Compared to the case of loss aversion only as in the previous subsection, endogeneity of the reference level has the *reinforcing* effect that the agent gives up even more upward potential in then-current consumption to guarantee consumption above the reference level. At the same time, the agent is also willing to accept somewhat larger consumption losses if the state of the economy is really adverse.

Figure 10 illustrates the optimal portfolio choice (i.e., the total dollar amount invested in the risky stock expressed as a percentage of the agent's initial wealth W_0) at age 70 as a function of the then-current log state price density for the case of risk-averse behavior in the loss domain. As in the case of loss aversion only, the optimal portfolio profile is U-shaped. While the then-current reference level affects the optimal consumption profile (see Figure 9), it does not impact the optimal portfolio profile. However, because of endogenous updating, optimal required wealth \widetilde{W}_t^* (i.e., wealth required to finance future optimal reference levels) is

Figure 8.
Gradual adjustment to financial shocks



Notes: The figure illustrates the impact of a positive shock in initial wealth on median consumption (expressed as a percentage of the agent’s initial wealth W_0) for the case of risk-averse behavior in the loss domain (i.e., $\gamma_1 = 1.2$). The curvature parameter γ_2 is set equal to 0.7 as before, and L to θ_0 . The right panel displays the case of no endogenous updating (i.e., $\alpha = \beta = 0\%$), while the left panel presents the case in which the agent endogenously updates the reference level over time (i.e., $\alpha = \beta = 20\%$). The dash-dotted lines in both panels represent the agent’s median consumption choice with a shock in initial wealth from 500 to 750 ($\times 1,000$ dollars) units.

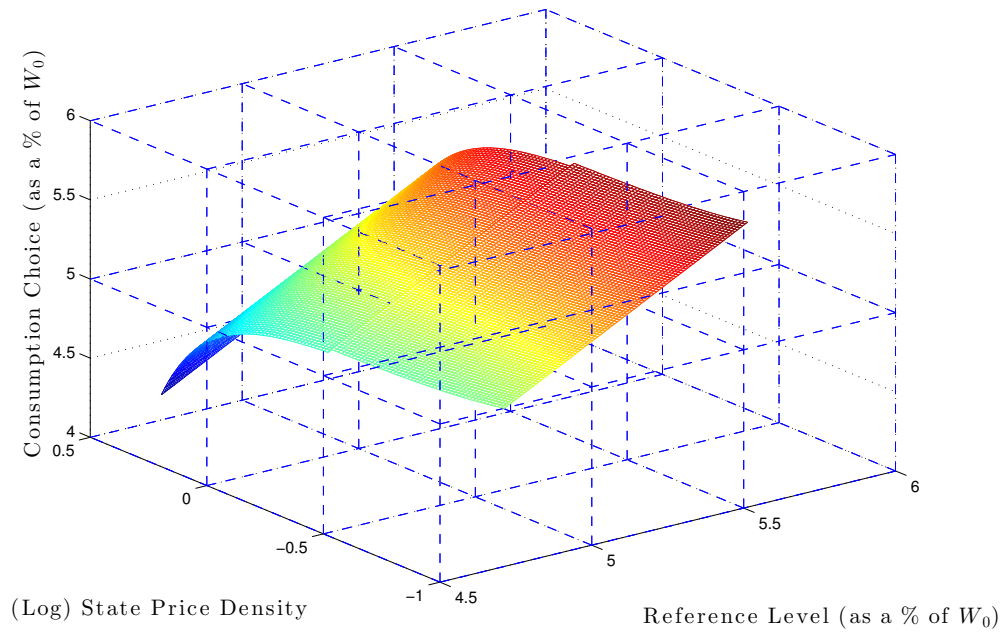
partly invested in the risky stock. Put differently, the dual portfolio choice no longer coincides with the agent’s optimal (primal) portfolio choice. By contrast, in the case of no endogenous updating as in the previous subsection, optimal required wealth \widetilde{W}_t^* is fully invested in the money market account. Since the reference level depends on the agent’s own past consumption choices (i.e., the reference level is path-dependent), the agent typically invests more in the risky stock under endogenous updating.

Figure 11 illustrates the median optimal portfolio choice measured as a fraction of total wealth invested in the risky stock as a function of the horizon, which represents the number of years spent in retirement. We observe that the agent implements a life cycle investment strategy (i.e., the fraction of wealth invested in the risky stock, on average, decreases as the agent ages). Indeed, since the agent has less time to absorb financial shocks as he grows older, the equity risk exposure, on average, decreases over the life cycle.²²

²²The slight increase in median optimal portfolio choice towards the end of the life span can be explained from the fact that the median optimal dual portfolio choice, which dictates the median optimal (primal) portfolio choice, displays a U-shaped pattern as a function of the horizon. This, in turn, is due to the fact that the absolute difference between optimal median consumption and the reference level as a function of the horizon is U-shaped, being smaller for intermediate horizons than for large and small horizons.

Figure 9.

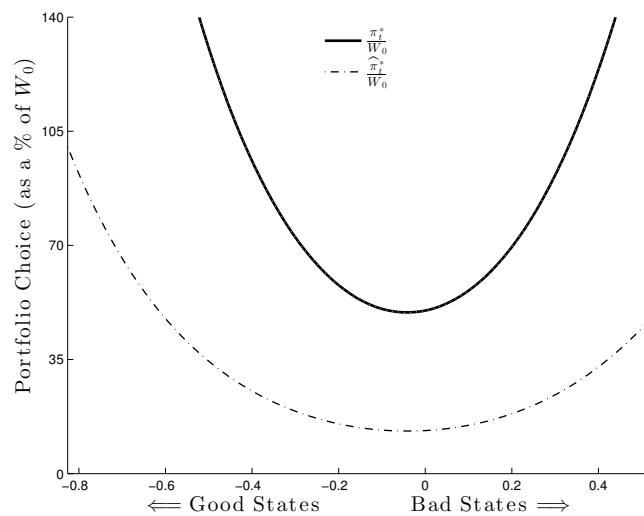
Consumption profile for the case of risk-averse behavior in the loss domain



Notes: The figure illustrates the optimal consumption choice (expressed as a percentage of the agent's initial wealth W_0) at age 70 as a function of the then-current log state price density and the then-current reference level. The curvature parameter γ_1 (γ_2) is set equal to 1.2 (0.7).

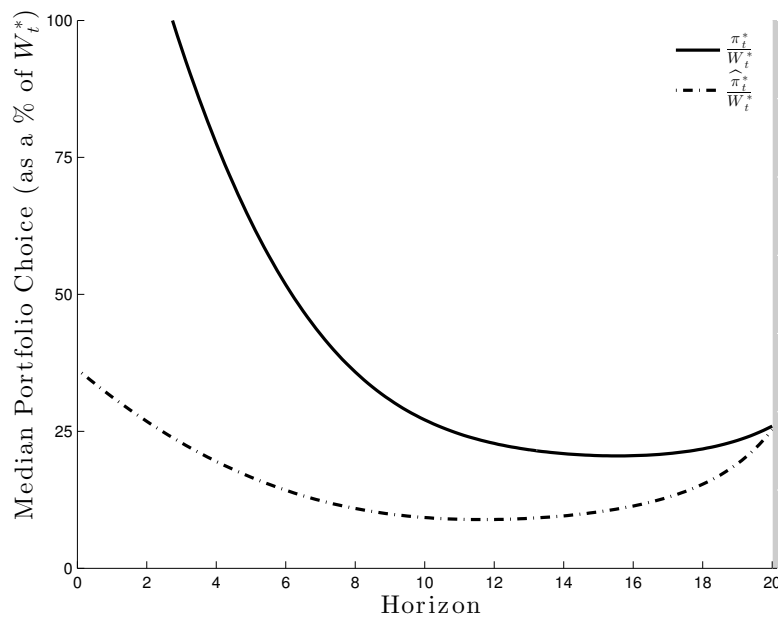
Figure 10.

Portfolio profile for the case of risk-averse behavior in the loss domain



Notes: The figure illustrates the optimal portfolio choice (i.e., the total dollar amount invested in the risky stock) at age 70 as a function of the then-current log state price density. The curvature parameter γ_1 (γ_2) is set equal to 1.2 (0.7). The dash-dotted line represents the optimal dual portfolio choice $\hat{\pi}_t^*/W_0$.

Figure 11.
Median portfolio choice



Notes: The figure illustrates the median optimal portfolio choice measured as a fraction of total wealth invested in the risky stock as a function of the horizon. The curvature parameter γ_1 (γ_2) is set equal to 1.2 (0.7). The dash-dotted line represents the optimal dual portfolio choice $\hat{\pi}_t^*/W_0$.

5.3 Welfare Analysis

This section conducts a welfare analysis. Section 5.3.1 reports the welfare losses (in terms of the relative decline in certainty equivalent consumption) associated with incorrect values of the agent’s preference parameters.²³ Precisely, we compute the welfare losses due to implementing suboptimal consumption and portfolio strategies derived by solving the agent’s maximization problem on the basis of wrong values of the loss aversion index κ , the depreciation parameter α and the endogeneity parameter β . Section 5.3.2 reports the welfare losses associated with implementing alternative (simpler) consumption and portfolio strategies. Throughout the welfare analysis, we assume that the agent’s *optimal* consumption and portfolio choice is characterized by the following (“true”) values of the preference parameters: $\theta_0 = 5.5\% \cdot W_0$, $\kappa = 2.5$, $\alpha = \beta = 20\%$, $\gamma_1 = 1.2$ and $\gamma_2 = 0.7$. Thus, the agent is risk-averse in the loss domain. The welfare losses are computed *relative* to the agent’s optimal consumption and portfolio strategy. The Appendix outlines the numerical procedure employed to compute the welfare losses. This procedure is non-standard due to the endogeneity of the reference level. The numerical procedure is implemented with $\Delta t = 1/8$ and $\mathcal{S} = 1,000,000$. Here, Δt denotes the time step and \mathcal{S} represents the total number of simulations.

5.3.1 Welfare Losses Due to Incorrect Parameter Values

Tables 2 and 3 report the welfare losses associated with implementing suboptimal consumption and portfolio strategies derived on the basis of wrong values of the loss aversion index κ , the depreciation parameter α and the endogeneity parameter β . In Table 2 we assume that the agent’s initial surplus wealth is equal to zero, while in Table 3 we assume that the agent has positive initial surplus wealth.²⁴ Table 2 shows that the welfare losses associated with incorrectly assuming a constant reference level (i.e., $\alpha = \beta = 0\%$) are substantial. Specifically, the welfare loss is about 30%. If the agent has positive initial surplus wealth, as in Table 3, this welfare loss is even larger. More generally, the tables reveal that consumption and portfolio strategies based on a constant exogenous reference level or on a very limited degree of endogeneity, thus implying no or only very limited smoothing of financial shocks, substantially reduce welfare. At the same time we observe that the impact of a change in the loss aversion index κ is larger when the agent’s initial surplus wealth is equal to zero than when the agent’s initial surplus wealth is positive. Indeed, κ determines the multiplicity of states in which consumption falls below the reference level. As a consequence, the impact of a change in κ is more pronounced when initial surplus wealth is small.

²³We define the certainty equivalent of an uncertain consumption strategy to be the constant, certain consumption level that yields indifference to the uncertain consumption strategy.

²⁴More specifically, Table 2 assumes that the agent’s initial wealth W_0 equals 500 ($\times 1,000$ dollars) units, while Table 3 assumes that W_0 is equal to 750 ($\times 1,000$ dollars) units.

Table 2.

Welfare losses due to incorrect parameter values (zero initial surplus wealth)

Loss aversion index (κ)	Endogeneity parameter (β)			
	0	0.05	0.10	0.20
2.5	31.93	17.50	8.25	0
5	28.58	22.56	18.33	11.85
10	28.00	25.99	24.87	22.40

Notes: The table reports the welfare losses (in terms of the relative decline in certainty equivalent consumption) due to implementing suboptimal consumption and portfolio strategies derived on the basis of wrong values of the loss aversion index κ , the depreciation parameter α , and the endogeneity parameter β . The depreciation parameter α always equals the endogeneity parameter β . The agent has zero initial surplus wealth. The numbers represent a percentage.

Table 3.

Welfare losses due to incorrect parameter values (positive initial surplus wealth)

Loss Aversion Index (κ)	Endogeneity Parameter (β)			
	0	0.05	0.10	0.20
2.5	89.33	13.32	0.93	0
5	88.97	13.22	1.14	0.42
10	88.86	13.20	1.17	0.51

Notes: The table reports the welfare losses (in terms of the relative decline in certainty equivalent consumption) due to implementing suboptimal consumption and portfolio strategies derived on the basis of wrong values of the loss aversion index κ , the depreciation parameter α , and the endogeneity parameter β . The depreciation parameter α always equals the endogeneity parameter β . The agent has positive initial surplus wealth. The numbers represent a percentage.

5.3.2 Welfare Losses Due to Alternative Strategies

Table 4 reports the welfare losses, compared to the optimal strategies of a loss averse agent who endogenously updates his reference level, due to implementing the consumption and portfolio strategy of an agent with CRRA utility (i.e., the Merton strategy). The welfare losses are reported for various values of the coefficient of relative risk aversion γ underlying the Merton strategy. The implementation of the Merton strategy, under which log consumption varies linearly with the log state price density and financial shocks are directly absorbed into future consumption, leads to substantial welfare losses of about 40%. The welfare losses are minimal for intermediate values of γ ($\gamma = 5$ in the table). We note that $\gamma = \infty$ corresponds to a risk-free strategy.

Finally, we consider the following practical consumption and portfolio strategy: we assume that the agent consumes a fraction $1/(T - t)$ of wealth W_t . Furthermore, we assume that a constant fraction of wealth is invested in the risky stock (i.e., we assume π_t/W_t to be constant), as under the Merton strategy. Table 5 reports the welfare losses for various values of the fraction of wealth invested in the risky stock. We observe that the welfare losses are again substantial,

Table 4.

Welfare losses due to implementing the Merton strategy

Relative Risk Aversion Coefficient (γ)				
1	2	5	10	∞
44.11	37.47	37.39	38.87	40.11

Notes: The table reports the welfare losses (in terms of the relative decline in certainty equivalent consumption) due to implementing the consumption and portfolio strategy of an agent with CRRA utility (i.e., the Merton strategy). The table reports the welfare losses for various values of the coefficient of relative risk aversion γ underlying the Merton strategy. The agent has zero initial surplus wealth. The numbers represent a percentage.

but smaller than when implementing the Merton consumption rule. Indeed, our numerical results reveal that the Merton strategy generates a more volatile consumption profile, with consumption falling below the reference level more often than when implementing the $1/(T-t)$ consumption rule. Thus, from the perspective of a loss averse agent, who strongly prefers to maintain consumption above the reference level, the $1/(T-t)$ consumption rule is less suboptimal than the Merton consumption rule. Furthermore, the welfare losses in Table 5 are relatively insensitive to changes in π_t/W_t . The welfare losses are minimal for relatively low fractions of wealth invested in the risky stock ($\pi_t/W_t = 10\%$ in the table). We also computed, under the $1/(T-t)$ consumption rule, the welfare losses associated with implementing various state-independent life cycle investment strategies. We find that the welfare losses do not substantially reduce when implementing a state-independent life cycle investment strategy.

Table 5.

Welfare losses due to implementing a practical alternative consumption and portfolio strategy

Fraction of Wealth Invested in the Risky Stock				
0	0.10	0.20	0.30	0.40
30.03	28.71	28.73	29.64	30.90

Notes: The table reports the welfare losses (in terms of the relative decline in certainty equivalent consumption) due to implementing a practical alternative consumption and portfolio strategy. The table reports the welfare losses for various values of the fraction of wealth invested in the risky stock (i.e., π_t/W_t). The agent has zero initial surplus wealth. The numbers represent a percentage.

6 An Alternative Utility Function

This section explores, as a robustness check, the agent's optimal consumption and portfolio choice under an alternative specification of the agent's instantaneous utility function. More specifically, we assume that the agent's utility function is represented by the kinked HARA utility function. The kinked HARA utility function emerges as a special case of (3.3) if (i) classical consumption utility m is represented by the HARA utility function and (ii) the gain-loss

utility function w equals the two-part power utility function v with $\gamma_1 = \gamma_2 = 1$. The HARA classical consumption utility function is defined as follows:²⁵

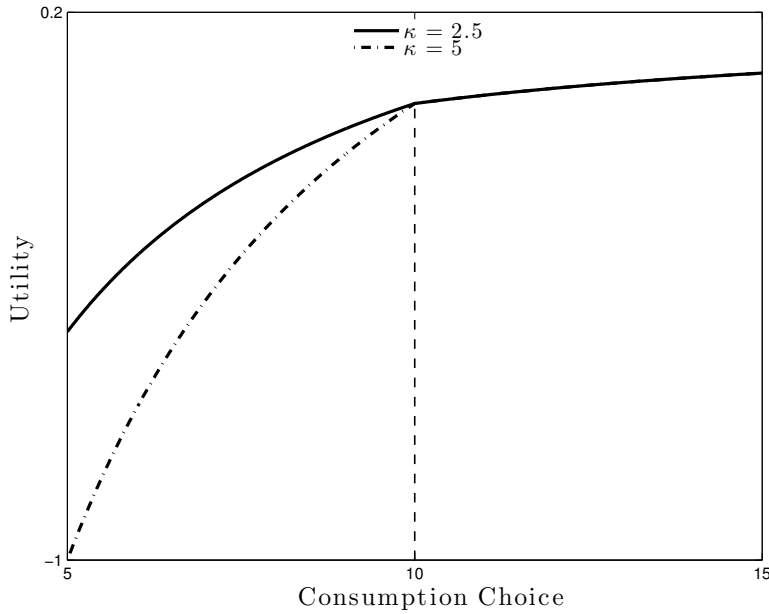
$$m(c_t) = \frac{\varphi}{1-\varphi} \left(\frac{\rho}{\varphi} c_t + \psi \right)^{1-\varphi}.$$

Here, $\varphi \in (0, \infty) \setminus \{1\}$, $\rho > 0$ and $\psi \geq 0$ are preference parameters.

Figure 12 illustrates the kinked CRRA utility function, which appears as a special case when $\rho = \varphi$ and $\psi = 0$, for $\kappa = 2.5$ and $\kappa = 5$. The figure shows that the kinked CRRA utility function has a kink at the reference level, with the slope of the utility function over losses being steeper than the slope of the utility function over gains. Furthermore, we observe that the kinked CRRA utility function is concave everywhere. Hence, the agent exhibits risk averse behavior in both the gain and the loss domain.

Figure 12.

The kinked CRRA utility function



Notes: The figure illustrates the kinked CRRA utility function (i.e., $\rho = \varphi$ and $\psi = 0$) for $\kappa = 2.5$ (solid line) and $\kappa = 5$ (dash-dotted line). The reference level θ_t is set equal to 10, the weight parameter η to 0 and the curvature parameter φ to 5.

Unfortunately, the kinked HARA utility function cannot be expressed in terms of the agent's surplus consumption choice $\hat{c}_t \equiv c_t - \theta_t$. As a direct consequence, the solution technique of [Schroder and Skiadas \(2002\)](#) is not applicable here. However, we can still obtain an analytical solution to the optimal consumption and portfolio choice problem if the agent's reference level

²⁵The HARA class of utility functions contains several important special cases. With suitable choice of preference parameters, the HARA utility function can exhibit increasing, decreasing or constant relative risk aversion. Important special cases are the commonly used CRRA ($\rho = \varphi$ and $\psi = 0$), exponential ($\psi = 1$ and $\varphi \rightarrow \infty$) and logarithmic ($\rho = 1$ and $\varphi \rightarrow 1$) utility functions.

is exogenously given. The assumption of an exogenous reference level implies that the agent's *own* (past) consumption choices do not affect the reference level. However, factors beyond the control of the agent are allowed to influence the reference level. Hence, the consumption and portfolio choice model considered in this section can be viewed as an external, rather than an internal, habit formation model (see, e.g., [Abel, 1990](#)). In what follows, the reader should keep in mind that the reference level is independent of the agent's own (past) consumption choices.

Theorem 3 below presents the optimal consumption choice for an agent with the kinked HARA utility function.

Theorem 3. *Consider an agent with the kinked HARA utility function and an exogenously given reference level process θ who solves the consumption and portfolio choice problem, with consumption constrained to be non-negative. Then the optimal consumption c_t^* at time $t \in [0, T]$ is given by*

$$c_t^* = \begin{cases} \frac{\varphi}{\rho} \left(\frac{y \exp\{\delta t\} M_t}{\rho} \right)^{-\frac{1}{\varphi}} - \frac{\psi\varphi}{\rho}, & \text{if } M_t < \underline{\xi}_t; \\ \theta_t, & \text{if } \underline{\xi}_t \leq M_t \leq \bar{\xi}_t; \\ \left[\frac{\varphi}{\rho} \left(\frac{y \exp\{\delta t\} M_t}{\rho \bar{\kappa}} \right)^{-\frac{1}{\varphi}} - \frac{\psi\varphi}{\rho} \right] \vee 0, & \text{if } M_t > \bar{\xi}_t. \end{cases}$$

Here, $\bar{\kappa} \equiv \eta + (1 - \eta)\kappa$ stands for the adjusted loss aversion index. The thresholds $\underline{\xi}_t$ and $\bar{\xi}_t$ are defined as follows:

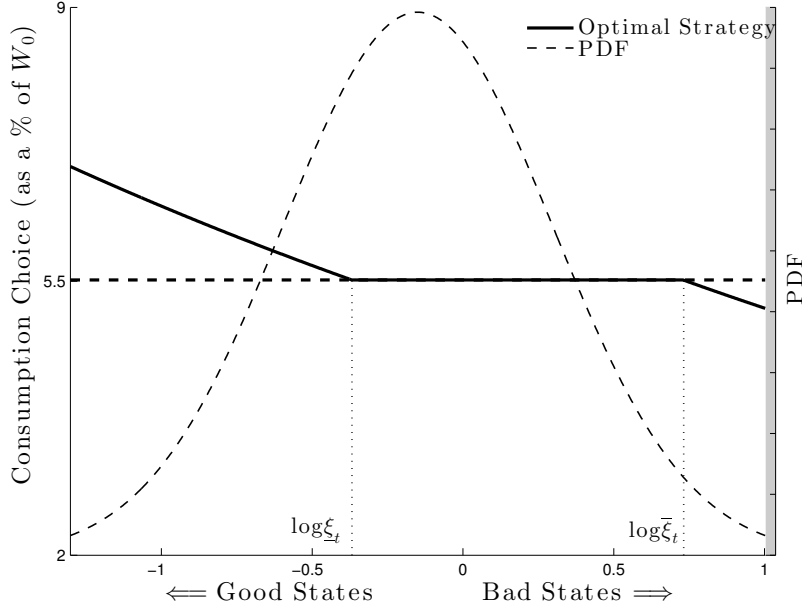
$$\underline{\xi}_t = \frac{\rho}{y} \exp\{-\delta t\} \left(\frac{\xi}{\varphi} \theta_t + \psi \right)^{-\varphi}, \quad \bar{\xi}_t = \frac{\rho \bar{\kappa}}{y} \exp\{-\delta t\} \left(\frac{\xi}{\varphi} \theta_t + \psi \right)^{-\varphi}.$$

The Lagrange multiplier y is chosen such that the static budget constraint holds with equality.

Theorem 3 shows that the state price density can be divided into three regions. In good economic scenarios (i.e., low state prices), consumption is (strictly) larger than the reference level; in these scenarios, the agent can afford to consume above the reference level. Next, in intermediate economic scenarios (i.e., intermediate state prices), consumption is equal to the reference level. The adjusted loss aversion index $\bar{\kappa}$ determines the multiplicity of states in which consumption is equal to the reference level. Finally, in bad economic scenarios (i.e., high state prices), the agent's wealth is insufficient to finance consumption at the reference level. In the case of two-part power utility (see Section 4), similarly, the optimal consumption choice also falls below the reference level in bad states of the world. Figure 13 illustrates the optimal consumption profile of an agent with kinked CRRA utility. We observe that, as before, the optimal consumption choice as a function of the log state price density is 90° rotated S-shaped, thus confirming the impact of loss aversion on the optimal consumption profile. We also observe that c_t^* is a continuous function of the state price density. In particular, the optimal consumption profile does not exhibit a jump at the reference level. Indeed, marginal utility at the reference level is finite.

Figure 13.

Optimal consumption profile of an agent with kinked CRRA utility



Notes: The figure shows the optimal consumption choice (expressed as a percentage of the agent's initial wealth W_0) at age 70 as a function of the then-current log state price density. The curvature parameter φ is set equal to 4. The remaining parameter values are the same as in Section 5. The dashed line corresponds to the agent's reference level (expressed as a percentage of W_0). The dotted line shows the probability density function (PDF) of the then-current log state price density conditional upon information available at the age of retirement.

The agent's optimal wealth W_t^* can be decomposed in the same way as in Section 4:

$$W_t^* = \widehat{W}_t^* + \widetilde{W}_t^* = \widehat{W}_t^{G*} + \widehat{W}_t^{L*} + \widetilde{W}_t^*$$

Proposition 3 below presents \widehat{W}_t^{G*} , \widehat{W}_t^{L*} and \widetilde{W}_t^* for the case of a constant investment opportunity set (i.e. $r_t = r$, $\sigma_t = \sigma$ and $\lambda_t = \lambda$).

Proposition 3. *Consider an agent with the kinked HARA utility function and an exogenously given reference level process θ who solves the consumption and portfolio choice problem, with consumption constrained to be non-negative and assuming a constant investment opportunity set. Let \mathcal{N} be the cumulative distribution function of a standard normal random variable. Define C , $d_1(x)$ and $d_2(x)$ as follows:*

$$C \equiv \frac{\delta + r(\varphi - 1)}{\varphi} + \frac{1}{2} \frac{\varphi - 1}{\varphi^2} \|\lambda\|^2,$$

$$d_1(x) = \frac{1}{\|\lambda\| \sqrt{s-t}} \left[\log(x) - \log(M_t) + \left(r - \frac{1}{2} \|\lambda\|^2 \right) (s-t) \right],$$

$$d_2(x) = d_1(x) + \frac{\|\lambda\|}{\varphi} \sqrt{s-t}.$$

Then:

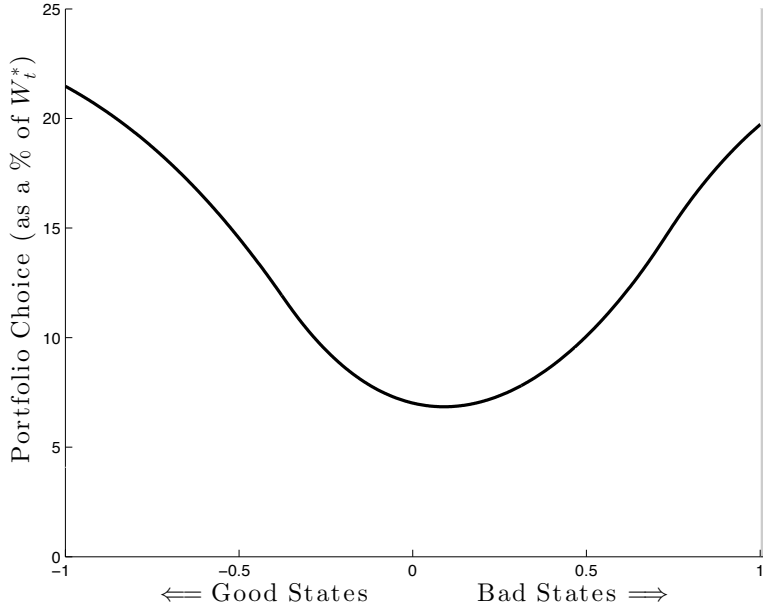
$$\begin{aligned}\widehat{W}_t^{G*} &= \frac{\varphi}{\rho} \left(\frac{y \exp\{\delta t\}}{\rho} \right)^{-\frac{1}{\varphi}} M_t^{-\frac{1}{\varphi}} \int_t^T \exp\{-C(s-t)\} \mathcal{N}[d_2(\underline{\xi}_s)] ds \\ &\quad - \frac{\psi\varphi}{\rho} \int_t^T \exp\{-r(s-t)\} \mathcal{N}[d_1(\underline{\xi}_s)] ds, \\ \widehat{W}_t^{L*} &= \frac{\varphi}{\rho} \left(\frac{y \exp\{\delta t\}}{\rho\bar{\kappa}} \right)^{-\frac{1}{\varphi}} M_t^{-\frac{1}{\varphi}} \int_t^T \exp\{-r(s-t)\} (\mathcal{N}[d_1(\bar{\xi}_s^*)] - \mathcal{N}[d_1(\bar{\xi}_s)]) ds \\ &\quad - \frac{\psi\varphi}{\rho} \int_t^T \exp\{-r(s-t)\} (\mathcal{N}[d_1(\bar{\xi}_s^*)] - \mathcal{N}[d_1(\bar{\xi}_s)]) ds, \\ \widetilde{W}_t^* &= \int_t^T \theta_s \exp\{-r(s-t)\} (\mathcal{N}[d_1(\bar{\xi}_s)] - \mathcal{N}[d_1(\underline{\xi}_s)]) ds.\end{aligned}$$

Here, $\bar{\xi}_s^* \equiv \frac{\psi^{-\varphi} \rho \bar{\kappa}}{y \exp\{\delta t\}}$.

The agent's optimal portfolio choice π_t^* can be computed in a similar way as in Section 4. Figure 14 illustrates the optimal portfolio profile of an agent with kinked CRRA utility. We observe that the optimal portfolio profile displays again a U-shaped pattern. In good as well as in bad states, the agent behaves like a CRRA agent. In particular, in these states, the fraction of wealth invested in the risky stock is equal to the constant $\lambda/(\sigma\varphi)$.

Figure 14.

Optimal portfolio profile of an agent with kinked CRRA utility



Notes: The figure shows the optimal portfolio choice measured as a fraction of total wealth invested in the risky stock at age 70 as a function of the then-current log state price density. The curvature parameter φ is set equal to 4. The remaining parameter values are the same as in Section 5.

7 Conclusion

We have explicitly derived the optimal consumption and portfolio choice under the two-part power utility function of [Tversky and Kahneman \(1992\)](#) while allowing the agent to endogenously update his reference level over time. We have shown that the loss averse feature of the utility function gives rise to a nonlinear consumption profile, inducing a (soft) guarantee on consumption, and that endogenous updating of the reference level implies smoothing of financial shocks.

We have assumed that agents are able to objectively evaluate the probabilities associated with future outcomes. A large body of research suggests that agents subjectively weight probabilities and e.g., have a tendency to overweight unlikely extreme outcomes (see, e.g., [Abdellaoui, 2000](#)). [Jin and Zhou \(2008\)](#) and [He and Zhou \(2011, 2014\)](#) consider optimal portfolio choice under subjective probability weighting; see also [Laeven and Stadjé \(2014\)](#). However, these authors do not consider intertemporal consumption or endogenous updating of the reference level. In future work we intend to extend our setting with intertemporal consumption and endogenous updating of the reference level to explore the impact of probability weighting on the optimal consumption and portfolio choice. Interestingly, as already shown by [He and Zhou \(2014\)](#), probability weighting may generate an endogenous insurance if small probabilities are sufficiently overweighted.

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Appendix

The Dual Technique

Schroder and Skiadas (2002) show that a generic consumption and portfolio choice model with linear internal habit formation can be mechanically transformed into a *dual* consumption and portfolio choice model without linear internal habit formation. The dual technique can be applied to an arbitrary utility function, including the two-part power utility function v [see expression (3.1)]. To formulate the dual consumption and portfolio choice model, let us define the agent's *surplus consumption choice* \widehat{c}_t as the agent's consumption choice c_t minus the agent's reference level θ_t ; that is, $\widehat{c}_t \equiv c_t - \theta_t$. We can view \widehat{c} as a gain process.²⁶ The agent's maximization problem (4.1) is now equivalent to the following dual problem:

$$\begin{aligned} & \underset{\widehat{c}}{\text{maximize}} && \mathbb{E} \left[\int_0^T \exp \{-\delta t\} v(\widehat{c}_t) dt \right] \\ & \text{subject to} && \mathbb{E} \left[\int_0^T \widehat{M}_t \widehat{c}_t dt \right] \leq \widehat{W}_0 (1 + \beta A_0), \quad \widehat{c}_t \geq -L_t \quad \text{for all } t \in [0, T]. \end{aligned} \tag{A1}$$

Here, \widehat{M}_t and \widehat{W}_0 represent the dual counterparts of the state price density M_t and the agent's initial wealth W_0 , respectively.

The relationship between the agent's maximization problem (4.1) and the dual problem (A1) is characterized in terms of the auxiliary process A :

$$A_t \equiv \frac{1}{M_t} \mathbb{E}_t \left[\int_t^T M_s \exp \{-(\alpha - \beta)(s - t)\} ds \right].$$

We can view A_t as the time t price of a bond paying a continuous coupon. In case the investment opportunity set is constant, A_t only depends on time t . As a direct consequence, the optimal portfolio choice can be computed explicitly in this case. The dual state price density \widehat{M}_t and the dual initial wealth \widehat{W}_0 are given by

$$\widehat{M}_t \equiv M_t (1 + \beta A_t), \quad \widehat{W}_0 \equiv \frac{W_0 - A_0 \theta_0}{1 + \beta A_0}.$$

Furthermore, the dual reference level

$$\widehat{\theta}_s = \beta \int_t^s \exp \{-(\alpha - \beta)(s - u)\} \widehat{c}_u du + \exp \{-(\alpha - \beta)(s - t)\} \widehat{\theta}_t, \quad s \geq t \geq 0,$$

is equal to the agent's reference level θ_s .

Surplus wealth \widehat{W}_t is defined as follows:

$$\widehat{W}_t \equiv \frac{1}{\widehat{M}_t} \mathbb{E}_t \left[\int_t^T \widehat{M}_s \widehat{c}_s ds \right].$$

²⁶We note that a negative gain corresponds to a (positive) loss.

Surplus wealth \widehat{W}_t is invested in a dual financial market that is characterized by the dual risk-free rate \widehat{r}_t , the dual volatility $\widehat{\sigma}_t$ and the dual market price of risk $\widehat{\lambda}_t$:

$$\widehat{r}_t \equiv \beta + \frac{r_t - \alpha\beta A_t}{1 + \beta A_t}, \quad \widehat{\sigma}_t \equiv \sigma_t,$$

$$\widehat{\lambda}_t \equiv \lambda_t - \frac{\beta}{1 + \beta A_t} \int_t^T \exp\{-(\alpha - \beta)(s - t)\} P_{t,s} \Psi_{t,s} ds.$$

Here, $P_{t,s}$ corresponds to the time t price of a default-free unit discount bond that matures at time $s \geq t$ and $\Psi_{t,s}$ stands for the time t volatility of the instantaneous return on such a bond (all in the primal financial market). The optimal dual portfolio choice $\widehat{\pi}_t^*$ is determined such that it finances the optimal surplus consumption choice \widehat{c}_t^* .

The next proposition is adapted from [Schroder and Skiadas \(2002\)](#).

Proposition 4. *Suppose that we have solved the dual problem (A1). Let us denote the optimal surplus consumption choice by \widehat{c}_t^* , the optimal dual reference level by $\widehat{\theta}_t^*$, the optimal surplus wealth by \widehat{W}_t^* and the optimal dual portfolio choice by $\widehat{\pi}_t^*$. Then:*

- *The optimal consumption for the agent at time $0 \leq t \leq T$ is given by*

$$c_t^* = \widehat{c}_t^* + \widehat{\theta}_t^*.$$

- *The optimal wealth for the agent at time $0 \leq t \leq T$ is given by*

$$W_t^* = \widehat{W}_t^* + \beta A_t \widehat{W}_t^* + A_t \widehat{\theta}_t^*.$$

- *The optimal portfolio choice for the agent at time $0 \leq t \leq T$ is given by*

$$\pi_t^* = \widehat{\pi}_t^* + \beta A_t \widehat{\pi}_t^* + \left(\beta \widehat{W}_t^* + \widehat{\theta}_t^* \right) (\widehat{\sigma}_t)^{-1} \int_t^T \exp\{-(\alpha - \beta)(s - t)\} P_{t,s} \Psi_{t,s} ds.$$

Proposition 4 shows how to transform the optimal solution to the dual problem (A1) back into the optimal solution to the agent's maximization problem (4.1).

Proof of Theorem 1

The proof uses some of the techniques developed by [Basak and Shapiro \(2001\)](#) and [Berkelaar et al. \(2004\)](#) to deal with pseudo-concavity and non-differentiability aspects of the problem and adapts these to our setting with intertemporal consumption.

The dual problem, equivalent to the agent's maximization problem (4.1), is given by

$$\begin{aligned} & \underset{\widehat{c}}{\text{maximize}} && \mathbb{E} \left[\int_0^T \exp\{-\delta t\} v(\widehat{c}_t) dt \right] \\ & \text{subject to} && \mathbb{E} \left[\int_0^T \widehat{M}_t \widehat{c}_t dt \right] \leq \widehat{W}_0 (1 + \beta A_0), \quad \widehat{c}_t \geq -L_t \quad \text{for all } t \in [0, T]. \end{aligned}$$

The corresponding Lagrangian \mathcal{L} is defined as follows:

$$\begin{aligned}\mathcal{L} &= \mathbb{E} \left[\int_0^T \exp \{-\delta t\} v(\widehat{c}_t) dt \right] + y \left(\widehat{W}_0 - \mathbb{E} \left[\int_0^T \widehat{M}_t \widehat{c}_t dt \right] \right) \\ &= \int_0^T \mathbb{E} \left[\exp \{-\delta t\} v(\widehat{c}_t) - y \widehat{M}_t \widehat{c}_t \right] dt + y \widehat{W}_0.\end{aligned}$$

Here, y denotes the Lagrange multiplier associated with the static budget constraint. The agent wishes to maximize $\exp \{-\delta t\} v(\widehat{c}_t) - y \widehat{M}_t \widehat{c}_t$ subject to $\widehat{c}_t \geq -L_t$. Denote the part of the two-part power utility function with domain below zero by v_1 , and the part with domain above zero by v_2 . Let us denote by c_{1t}^* the agent's optimal surplus consumption choice for utility function v_1 , and by c_{2t}^* the agent's optimal surplus consumption choice for utility function v_2 .

We first consider the case where the agent is risk-averse in the loss domain. Due to the concavity of v_1 and v_2 , the optimal surplus consumption choices c_{1t}^* and c_{2t}^* satisfy the following optimality conditions:²⁷

$$\begin{aligned}\exp \{-\delta t\} v'_j(c_{jt}^*) &= y \widehat{M}_t - x_{jt}, & c_{jt}^* &\geq -L_t, & \text{for } j &= 1, 2, \\ x_{jt}(c_{jt}^* + L_t) &= 0, & x_{jt} &\geq 0, & \text{for } j &= 1, 2.\end{aligned}$$

Here, x_{jt} denotes the Lagrange multiplier associated with the constraint on surplus consumption. After solving the optimality conditions, we obtain the following two local maxima:

$$c_{1t}^* = - \left[\left(l_t \widehat{M}_t \right)^{\frac{1}{\gamma_1 - 1}} \wedge L_t \right], \quad c_{2t}^* = \left(k_t \widehat{M}_t \right)^{\frac{1}{\gamma_2 - 1}}.$$

Here, $l_t \equiv y \exp \{\delta t\} \cdot \frac{1}{\kappa \gamma_1}$ and $k_t \equiv y \exp \{\delta t\} \cdot \frac{1}{\gamma_2}$.

To determine the global maximum \widehat{c}_t^* , we introduce the following function:

$$\begin{aligned}f(\widehat{M}_t) &= \exp \{-\delta t\} v(c_{2t}^*) - y \widehat{M}_t c_{2t}^* - \left[\exp \{-\delta t\} v(c_{1t}^*) - y \widehat{M}_t c_{1t}^* \right] \\ &= \exp \{-\delta t\} (1 - \gamma_2) \left(k_t \widehat{M}_t \right)^{\frac{\gamma_2}{\gamma_2 - 1}} + \kappa \exp \{-\delta t\} \left[\left(l_t \widehat{M}_t \right)^{\frac{1}{\gamma_1 - 1}} \wedge L_t \right]^{\gamma_1} \\ &\quad - y \widehat{M}_t \left[\left(l_t \widehat{M}_t \right)^{\frac{1}{\gamma_1 - 1}} \wedge L_t \right].\end{aligned}$$

The global maximum \widehat{c}_t^* is equal to c_{2t}^* if $f(\widehat{M}_t) \geq 0$; and equals c_{1t}^* otherwise. It follows that $\lim_{\widehat{M}_t \rightarrow \infty} f(\widehat{M}_t) = -\infty$, $\lim_{\widehat{M}_t \rightarrow 0} f(\widehat{M}_t) = \infty$ and $f'(\widehat{M}_t) < 0$ for all \widehat{M}_t . Hence, $f(\widehat{M}_t)$ is strictly decreasing. As a direct consequence, $f(\widehat{M}_t)$ has one zero in the interval $(0, \infty)$. Define ξ_t to be such that $f(\xi_t) = 0$. The global maximum \widehat{c}_t^* is equal to c_{2t}^* if $\widehat{M}_t \leq \xi_t$; and equals c_{1t}^* otherwise.

We now consider the case where the agent is risk-seeking in the loss domain. Due to the concavity of v_2 , the optimal surplus consumption choice c_{2t}^* satisfies the following optimality

²⁷The derivative of a function f at a point a is denoted by $f'(a)$.

conditions:

$$\exp\{-\delta t\} v_2'(c_{jt}^*) = y\widehat{M}_t - x_{2t}, \quad c_{2t}^* \geq -L_t,$$

$$x_{2t}(c_{2t}^* + L_t) = 0, \quad x_{2t} \geq 0.$$

After solving the optimality conditions, we obtain the following local maximum:

$$c_{2t}^* = \left(k_t \widehat{M}_t\right)^{\frac{1}{\gamma_2 - 1}}.$$

Due to the convexity of v_1 , the optimal surplus consumption choice c_{1t}^* lies at a corner point of the feasible region. Hence, the only two possible candidates for c_{1t}^* are $-L_t$ and 0.

To determine the global maximum \widehat{c}_t^* , we introduce the following function:

$$g\left(\widehat{M}_t\right) = \exp\{-\delta t\} v(c_{2t}^*) - y\widehat{M}_t c_{2t}^* - \left[\exp\{-\delta t\} v(c_{1t}^*) - y\widehat{M}_t c_{1t}^*\right].$$

The global maximum \widehat{c}_t^* is equal to c_{2t}^* if $g\left(\widehat{M}_t\right) \geq 0$; and equals c_{1t}^* otherwise. We distinguish between the following two cases:

- $c_{1t}^* = 0$. Straightforward computations show that $g\left(\widehat{M}_t\right)$ is given by

$$g\left(\widehat{M}_t\right) = \exp\{-\delta t\} (1 - \gamma_2) \left(k_t \widehat{M}_t\right)^{\frac{\gamma_2}{\gamma_2 - 1}}.$$

Since $0 < \gamma_2 < 1$ and $y > 0$, it follows that $g\left(\widehat{M}_t\right) > 0$ for all \widehat{M}_t . We conclude that $c_{1t}^* = 0$ is never optimal.

- $c_{1t}^* = -L_t$. Straightforward computations show that $g\left(\widehat{M}_t\right)$ is given by

$$g\left(\widehat{M}_t\right) = \exp\{-\delta t\} (1 - \gamma_2) \left(k_t \widehat{M}_t\right)^{\frac{\gamma_2}{\gamma_2 - 1}} + \exp\{-\delta t\} \kappa L_t^{\gamma_1} - y\widehat{M}_t L_t.$$

It follows that $g\left(\widehat{M}_t\right) > 0$ for all $\widehat{M}_t \leq \frac{\kappa}{y} \exp\{-\delta t\} L_t^{\gamma_1 - 1}$. Further, $\lim_{\widehat{M}_t \rightarrow \infty} g\left(\widehat{M}_t\right) = -\infty$ and $g'\left(\widehat{M}_t\right) < 0$ for all \widehat{M}_t . Hence, $g\left(\widehat{M}_t\right)$ is strictly decreasing. As a direct consequence, $g\left(\widehat{M}_t\right)$ has one zero in the interval $\left(\frac{\kappa}{y} \exp\{-\delta t\} L_t^{\gamma_1 - 1}, \infty\right)$. Define ξ_t to be such that $g\left(\xi_t\right) = 0$. It follows that the global maximum \widehat{c}_t^* is equal to c_{2t}^* if $\widehat{M}_t \leq \xi_t$; and equals c_{1t}^* otherwise.

A standard verification (see, e.g., [Karatzas and Shreve, 1998](#), p. 103) that the optimal solutions obtained from the Lagrangian are the optimal solutions to the dual problem completes the proof. ■

Proof of Proposition 1

We distinguish between the following two cases:

- Risk-averse behavior in the loss domain. Define the following function:

$$\tilde{f}(x) \equiv (1 - \gamma_2) \left(\frac{x}{\gamma_2} \right)^{\frac{\gamma_2}{\gamma_2-1}} + \kappa \left[\left(\frac{x}{\gamma_1 \kappa} \right)^{\frac{1}{\gamma_1-1}} \wedge L_t \right]^{\gamma_1} - x \left[\left(\frac{x}{\gamma_1 \kappa} \right)^{\frac{1}{\gamma_1-1}} \wedge L_t \right].$$

Let $\tilde{\xi}_t$ be such that $\tilde{f}(\tilde{\xi}_t) = 0$. It follows that $\tilde{\xi}_t = y \exp\{\delta t\} \xi_t$. The quantity $\tilde{\xi}_t$ increases as the loss aversion index κ increases. Furthermore, initial surplus wealth \widehat{W}_0 decreases with the initial reference level θ_0 , increases with the depreciation parameter α (provided that \widehat{W}_0 is non-negative), and decreases with the endogeneity parameter β (provided that \widehat{W}_0 is non-negative).

- Risk-seeking behavior in the loss domain. Define the following function:

$$\tilde{g}(x) \equiv (1 - \gamma_2) \left(\frac{x}{\gamma_2} \right)^{\frac{\gamma_2}{\gamma_2-1}} + \kappa L_t^{\gamma_1} - x L_t.$$

Let $\tilde{\xi}_t$ be such that $\tilde{g}(\tilde{\xi}_t) = 0$. It follows that $\tilde{\xi}_t = y \exp\{\delta t\} \xi_t$. The quantity $\tilde{\xi}_t$ increases as the loss aversion index κ increases. Furthermore, initial surplus wealth \widehat{W}_0 decreases with the initial reference level θ_0 , increases with the depreciation parameter α (provided that \widehat{W}_0 is non-negative), and decreases with the endogeneity parameter β (provided that \widehat{W}_0 is non-negative).

The proposition now follows straightforwardly from [Berkelaar et al. \(2004\)](#). ■

Proof of Proposition 2

Optimal surplus wealth is given by

$$\widehat{W}_t^* = \frac{1}{\widehat{M}_t} \mathbb{E}_t \left[\int_t^T \widehat{M}_s \widehat{c}_s^* ds \right]. \quad (\text{A2})$$

We first consider the case where the agent is risk-averse in the loss domain. Substituting the optimal surplus consumption choice \widehat{c}_s^* into equation (A2) yields

$$\begin{aligned} \widehat{W}_t^* &= \frac{1}{\widehat{M}_t} \mathbb{E}_t \left[\int_t^T \widehat{M}_s \left(k_s \widehat{M}_s \right)^{\frac{1}{\gamma_2-1}} \mathbb{1}_{[\widehat{M}_s \leq \xi_s]} ds \right. \\ &\quad \left. - \int_t^T \widehat{M}_s \left(l_s \widehat{M}_s \right)^{\frac{1}{\gamma_1-1}} \mathbb{1}_{[\xi_s < \widehat{M}_s < \xi_s \vee \zeta_s]} ds - \int_t^T \widehat{M}_s L_s \mathbb{1}_{[\widehat{M}_s \geq \xi_s \vee \zeta_s]} ds \right] \\ &= \left(k_t \widehat{M}_t \right)^{\frac{1}{\gamma_2-1}} \mathbb{E}_t \left[\int_t^T \left(\frac{\widehat{M}_s}{\widehat{M}_t} \right)^{\frac{\gamma_2}{\gamma_2-1}} \exp \left\{ \frac{\delta(s-t)}{\gamma_2-1} \right\} \mathbb{1}_{[\widehat{M}_s \leq \xi_s]} ds \right] \\ &\quad - \left(l_t \widehat{M}_t \right)^{\frac{1}{\gamma_1-1}} \mathbb{E}_t \left[\int_t^T \left(\frac{\widehat{M}_s}{\widehat{M}_t} \right)^{\frac{\gamma_1}{\gamma_1-1}} \exp \left\{ \frac{\delta(s-t)}{\gamma_1-1} \right\} \mathbb{1}_{[\xi_s < \widehat{M}_s < \xi_s \vee \zeta_s]} ds \right] \\ &\quad - \mathbb{E}_t \left[\int_t^T \frac{\widehat{M}_s}{\widehat{M}_t} L_s \mathbb{1}_{[\widehat{M}_s \geq \xi_s \vee \zeta_s]} ds \right]. \end{aligned} \quad (\text{A3})$$

Here, $\zeta_s \equiv \exp\{-\delta s\} \frac{\gamma_1 \kappa}{y} L_s^{\gamma_1 - 1}$. The closed-form expression for \widehat{W}_t^* can be determined by computing the conditional expectations. In case the investment opportunity set is constant, we find

$$\mathbb{E}_t \left[\frac{\widehat{M}_s}{\widehat{M}_t} L_s \mathbb{1}_{[\widehat{M}_s \geq \xi_s \vee \zeta_s]} \right] = \exp \left\{ - \int_t^s \widehat{r}_u \, du \right\} L_s \mathcal{N}[-d_1(\xi_s \vee \zeta_s)], \quad (\text{A4})$$

$$\mathbb{E}_t \left[\left(\frac{\widehat{M}_s}{\widehat{M}_t} \right)^{\frac{\gamma_2}{\gamma_2 - 1}} \exp \left\{ \frac{\delta(s-t)}{\gamma_2 - 1} \right\} \mathbb{1}_{[\widehat{M}_s \leq \xi_s]} \right] = \exp \left\{ - \int_t^s \Gamma_u \, du \right\} \mathcal{N}[d_2(\xi_s)], \quad (\text{A5})$$

$$\begin{aligned} \mathbb{E}_t \left[\left(\frac{\widehat{M}_s}{\widehat{M}_t} \right)^{\frac{\gamma_1}{\gamma_1 - 1}} \exp \left\{ \frac{\delta(s-t)}{\gamma_1 - 1} \right\} \mathbb{1}_{[\xi_s < \widehat{M}_s < \xi_s \vee \zeta_s]} \right] &= \exp \left\{ - \int_t^s \Pi_u \, du \right\} \\ &\times (\mathcal{N}[d_3(\xi_s \vee \zeta_s)] - \mathcal{N}[d_3(\xi_s)]). \end{aligned} \quad (\text{A6})$$

Here, \mathcal{N} is the cumulative distribution function of a standard normal random variable, and Γ_u , Π_u , $d_1(x)$, $d_2(x)$ and $d_3(x)$ are defined as follows:

$$\begin{aligned} \Gamma_u &\equiv \frac{\delta - \gamma_2 \widehat{r}_u}{1 - \gamma_2} - \frac{1}{2} \frac{\gamma_2}{(1 - \gamma_2)^2} \|\lambda\|^2, & \Pi_u &\equiv \frac{\delta - \gamma_1 \widehat{r}_u}{1 - \gamma_1} - \frac{1}{2} \frac{\gamma_1}{(1 - \gamma_1)^2} \|\lambda\|^2, \\ d_1(x) &\equiv \frac{1}{\|\lambda\| \sqrt{s-t}} \cdot \left[\log(x) - \log(\widehat{M}_t) + \int_t^s \widehat{r}_u \, du - \frac{1}{2} \|\lambda\|^2 (s-t) \right], \\ d_2(x) &\equiv d_1(x) + \frac{\|\lambda\|}{1 - \gamma_2} \sqrt{s-t}, & d_3(x) &\equiv d_1(x) + \frac{\|\lambda\|}{1 - \gamma_1} \sqrt{s-t}. \end{aligned}$$

Substituting the conditional expectations (A4), (A5) and (A6) into equation (A3) yields the optimal surplus wealth.

We now consider the case where the agent is risk-seeking in the loss domain. Substituting the optimal surplus consumption choice \widehat{c}_s^* into equation (A2) yields

$$\begin{aligned} \widehat{W}_t^* &= \frac{1}{\widehat{M}_t} \mathbb{E}_t \left[\int_t^T \widehat{M}_s \left(k_s \widehat{M}_s \right)^{\frac{1}{\gamma_2 - 1}} \mathbb{1}_{[\widehat{M}_s \leq \xi_s]} \, ds - \int_t^T \widehat{M}_s L_s \mathbb{1}_{[\widehat{M}_s > \xi_s]} \, ds \right] \\ &= \left(k_t \widehat{M}_t \right)^{\frac{1}{\gamma_2 - 1}} \mathbb{E}_t \left[\int_t^T \left(\frac{\widehat{M}_s}{\widehat{M}_t} \right)^{\frac{\gamma_2}{\gamma_2 - 1}} \exp \left\{ \frac{\delta(s-t)}{\gamma_2 - 1} \right\} \mathbb{1}_{[\widehat{M}_s \leq \xi_s]} \, ds \right] \\ &\quad - \mathbb{E}_t \left[\int_t^T \frac{\widehat{M}_s}{\widehat{M}_t} L_s \mathbb{1}_{[\widehat{M}_s > \xi_s]} \, ds \right]. \end{aligned} \quad (\text{A7})$$

The closed-form expression for \widehat{W}_t^* can be determined by computing the conditional expectations.

In case the investment opportunity set is constant, we find

$$\mathbb{E}_t \left[\frac{\widehat{M}_s}{\widehat{M}_t} L_s \mathbb{1}_{[\widehat{M}_s > \xi_s]} \right] = \exp \left\{ - \int_t^s \widehat{r}_u \, du \right\} L_s \mathcal{N}[-d_1(\xi_s)], \quad (\text{A8})$$

$$\mathbb{E}_t \left[\left(\frac{\widehat{M}_s}{\widehat{M}_t} \right)^{\frac{\gamma_2}{\gamma_2 - 1}} \exp \left\{ \frac{\delta(s-t)}{\gamma_2 - 1} \right\} \mathbb{1}_{[\widehat{M}_s \leq \xi_s]} \right] = \exp \left\{ - \int_t^s \Gamma_u \, du \right\} \mathcal{N}[d_2(\xi_s)]. \quad (\text{A9})$$

Substituting the conditional expectations (A8) and (A9) into equation (A7) yields the optimal surplus wealth. ■

Welfare Analysis

This appendix describes a numerical procedure for computing welfare losses. The numerical procedure is based on the assumptions that the investment opportunity set is constant and the agent can only invest in one risky stock. We introduce the following notation:

- Δt : time step;
- $t_n \equiv n\Delta t$ for $n = 0, \dots, \lfloor \frac{T}{\Delta t} \rfloor$;
- \mathcal{S} : total number of simulations.

The floor operator $\lfloor \cdot \rfloor$ rounds a number downward to its nearest integer.

To compute the welfare loss associated with a suboptimal consumption strategy c_t , we apply the following steps:

1. We generate \mathcal{S} trajectories of the pricing kernel:

$$M_{t_{n+1}}^s = M_{t_n}^s - rM_{t_n}^s \Delta t - \lambda M_{t_n}^s \sqrt{\Delta t} \epsilon_{t_n}^s, \quad n = 0, \dots, \left\lfloor \frac{T}{\Delta t} \right\rfloor, \quad s = 1, \dots, \mathcal{S}.$$

Here, $\epsilon_{t_n}^s$ is a standard normally distributed random variable.

2. We compute the optimal surplus consumption choice $\widehat{c}_{t_n}^{*s}$ for $n = 0, \dots, \lfloor \frac{T}{\Delta t} \rfloor$ and $s = 1, \dots, \mathcal{S}$. We note that the optimal surplus consumption choice $\widehat{c}_{t_n}^{*s}$ is a function of the dual state price density $\widehat{M}_{t_n}^s \equiv M_{t_n}^s (1 + \beta A_{t_n})$. Expected utility can now be approximated by

$$\mathbb{E} \left[\int_0^T \exp \{-\delta t\} v(\widehat{c}_t^*) \, dt \right] \approx \frac{1}{\mathcal{S}} \sum_{s=1}^{\mathcal{S}} \sum_{n=0}^{\lfloor \frac{T}{\Delta t} \rfloor} \exp \{-\delta t_n\} v(\widehat{c}_{t_n}^{*s}) \Delta t. \quad (\text{A10})$$

The right-hand side of equation (A10) is an approximation of $\mathbb{E} \left[\int_0^T \exp \{-\delta t\} v(c_t^* - \theta_t^*) \, dt \right]$.

3. We solve for certainty equivalent consumption ce^* :

$$\frac{1}{S} \sum_{s=1}^S \sum_{n=0}^{\lfloor \frac{T}{\Delta t} \rfloor} \exp\{-\delta t_n\} v(\widehat{c}_{t_n}^{*s}) \Delta t = \sum_{n=0}^{\lfloor \frac{T}{\Delta t} \rfloor} \exp\{-\delta t_n\} v(ce^* - \theta_{t_n}^*) \Delta t,$$

where

$$\theta_{t_n}^* = \theta_0 \exp\{-\alpha t_n\} + \beta \sum_{i=0}^{n-1} \exp\{-\alpha(t_n - t_i)\} ce^* \Delta t.$$

4. We compute the suboptimal consumption strategy $\widehat{c}_{t_n}^s \equiv c_{t_n}^s - \theta_{t_n}^s$ for $n = 0, \dots, \lfloor \frac{T}{\Delta t} \rfloor$ and $s = 1, \dots, S$. Expected utility can now be approximated by

$$\mathbb{E} \left[\int_0^T \exp\{-\delta t\} v(\widehat{c}_t) dt \right] \approx \frac{1}{S} \sum_{s=1}^S \sum_{n=0}^{\lfloor \frac{T}{\Delta t} \rfloor} \exp\{-\delta t_n\} v(\widehat{c}_{t_n}^s) \Delta t.$$

5. We solve for certainty equivalent consumption ce :

$$\frac{1}{S} \sum_{s=1}^S \sum_{n=0}^{\lfloor \frac{T}{\Delta t} \rfloor} \exp\{-\delta t_n\} v(\widehat{c}_{t_n}^s) \Delta t = \sum_{n=0}^{\lfloor \frac{T}{\Delta t} \rfloor} \exp\{-\delta t_n\} v(ce - \theta_{t_n}) \Delta t,$$

where

$$\theta_{t_n} = \theta_0 \exp\{-\alpha t_n\} + \beta \sum_{i=0}^{n-1} \exp\{-\alpha(t_n - t_i)\} ce \Delta t.$$

6. Finally, we compute the welfare loss WL :

$$WL = \frac{ce^* - ce}{ce^*}.$$

Proof of Theorem 3

The proof uses some of the techniques developed by [Basak and Shapiro \(2001\)](#) and [Berkelaar et al. \(2004\)](#) and adapts these to our setting with intertemporal consumption.

The agent's maximization problem is given by

$$\begin{aligned} & \underset{c}{\text{maximize}} && \mathbb{E} \left[\int_0^T \exp\{-\delta t\} u(c_t; \theta_t) dt \right] \\ & \text{subject to} && \mathbb{E} \left[\int_0^T M_t c_t dt \right] \leq W_0, \quad c_t \geq 0 \quad \text{for all } t \in [0, T]. \end{aligned}$$

The corresponding Lagrangian \mathcal{L} is defined as follows:

$$\begin{aligned}\mathcal{L} &= \mathbb{E} \left[\int_0^T \exp \{-\delta t\} u(c_t; \theta_t) dt \right] + y \left(W_0 - \mathbb{E} \left[\int_0^T M_t c_t dt \right] \right) \\ &= \int_0^T \mathbb{E} [\exp \{-\delta t\} u(c_t; \theta_t) - y M_t c_t] dt + y W_0.\end{aligned}$$

Here, y denotes the Lagrange multiplier associated with the static budget constraint. The agent wishes to maximize $\exp \{-\delta t\} u(c_t; \theta_t) - y M_t c_t$ subject to $c_t \geq 0$. Denote the part of the utility function with domain below zero by u_1 , and the part with domain above zero by u_2 . Let us denote by c_{1t}^* the agent's optimal consumption choice for utility function u_1 , and by c_{2t}^* the agent's optimal consumption choice for utility function u_2 .

Due to the concavity of u_1 and u_2 , the optimal consumption choices c_{1t}^* and c_{2t}^* satisfy the following optimality conditions:

$$\begin{aligned}\exp \{-\delta t\} u'_j(c_{jt}^*; \theta_t) &= y M_t - x_{jt}, \quad c_{jt}^* \geq 0, \quad \text{for } j = 1, 2, \\ x_{jt} c_{jt}^* &= 0, \quad x_{jt} \geq 0, \quad \text{for } j = 1, 2.\end{aligned}$$

Here, x_{jt} denotes the Lagrange multiplier associated with the non-negativity constraint on consumption. After solving the optimality conditions, we obtain the following two local maxima:

$$\begin{aligned}c_{1t}^* &= \min \left\{ \theta_t, \left[\frac{\varphi}{\rho} \left(\frac{y \exp \{\delta t\} M_t}{\rho \bar{\kappa}} \right)^{-\frac{1}{\varphi}} - \frac{\psi \varphi}{\rho} \right] \vee 0 \right\}, \\ c_{2t}^* &= \max \left\{ \theta_t, \frac{\varphi}{\rho} \left(\frac{y \exp \{\delta t\} M_t}{\rho} \right)^{-\frac{1}{\varphi}} - \frac{\psi \varphi}{\rho} \right\}.\end{aligned}$$

Here, $\bar{\kappa} \equiv \eta + (1 - \eta) \cdot \kappa$.

To determine the global maximum c_t^* , we introduce the following function:

$$f(M_t) = \exp \{-\delta t\} u(c_{2t}^*; \theta_t) - y M_t c_{2t}^* - [\exp \{-\delta t\} u(c_{1t}^*; \theta_t) - y M_t c_{1t}^*].$$

The global maximum is equal to c_{2t}^* if $f(M_t) \geq 0$; and equals c_{1t}^* otherwise. It follows that $f(M_t)$ changes sign at the points $\underline{\xi}_t = \frac{\rho}{y} \exp \{-\delta t\} \left(\frac{\rho}{\varphi} \theta_t + \psi \right)^{-\varphi}$ and $\bar{\xi}_t = \frac{\rho \bar{\kappa}}{y} \exp \{-\delta t\} \left(\frac{\rho}{\varphi} \theta_t + \psi \right)^{-\varphi}$.

We consider the following three cases:

- $\underline{\xi}_t \leq M_t \leq \bar{\xi}_t$. It follows that θ_t is the only candidate solution. We conclude that $c_t^* = \theta_t$ is the global maximum.
- $M_t > \bar{\xi}_t$. We compare the candidate solutions $c_{1t}^* = \left[\frac{\varphi}{\rho} \left(\frac{y \exp \{\delta t\} M_t}{\rho \bar{\kappa}} \right)^{-\frac{1}{\varphi}} - \frac{\psi \varphi}{\rho} \right] \vee 0$ and $c_{2t}^* = \theta_t$. Some straightforward computations show that $f(\bar{\xi}_t) = 0$, $f'(\bar{\xi}_t) = 0$ and $f''(M_t) < 0$ for all $M_t > \bar{\xi}_t$. Hence, $f(M_t) < 0$ for all $M_t > \bar{\xi}_t$. We conclude that $c_t^* = c_{1t}^*$ is the global maximum.
- $M_t < \underline{\xi}_t$. We compare the candidate solutions $c_{1t}^* = \theta_t$ and $c_{2t}^* = \frac{\varphi}{\rho} \left(\frac{y \exp \{\delta t\} M_t}{\rho} \right)^{-\frac{1}{\varphi}} - \frac{\psi \varphi}{\rho}$.

Some straightforward computations show that $f(\underline{\xi}_t) = 0$, $f'(\underline{\xi}_t) = 0$ and $f''(\underline{\xi}_t) > 0$ for all $M_t < \underline{\xi}_t$. Hence, $f(M_t) > 0$ for all $M_t < \underline{\xi}_t$. We conclude that $c_t^* = c_{2t}^*$ is the global maximum.

A standard verification (see, e.g., [Karatzas and Shreve, 1998](#), p. 103) that the optimal solution obtained from the Lagrangian is the optimal solution to the static maximization problem completes the proof. \blacksquare

Proof of Proposition 3

Optimal wealth is given by

$$W_t^* = \frac{1}{M_t} \mathbb{E}_t \left[\int_t^T M_s c_s^* ds \right]. \quad (\text{A11})$$

Substituting the optimal consumption choice into equation (A11) yields

$$\begin{aligned} W_t^* &= \frac{1}{M_t} \mathbb{E}_t \left[\int_t^T M_s \theta_s \mathbb{1}_{[\underline{\xi}_s \leq M_s \leq \bar{\xi}_s]} ds \right. \\ &\quad + \int_t^T M_s \left\{ \frac{\varphi}{\rho} \left(\frac{y \exp\{\delta s\} M_s}{\rho} \right)^{-\frac{1}{\varphi}} - \frac{\psi\varphi}{\rho} \right\} \mathbb{1}_{[M_s < \underline{\xi}_s]} ds \\ &\quad \left. + \int_t^T M_s \left\{ \frac{\varphi}{\rho} \left(\frac{y \exp\{\delta s\} M_s}{\rho \bar{\kappa}} \right)^{-\frac{1}{\varphi}} - \frac{\psi\varphi}{\rho} \right\} \mathbb{1}_{[\bar{\xi}_s < M_s < \bar{\xi}_s^*]} ds \right] \\ &= \mathbb{E}_t \left[\int_t^T \frac{M_s}{M_t} \theta_s \mathbb{1}_{[\underline{\xi}_s \leq M_s \leq \bar{\xi}_s]} ds \right] \\ &\quad + \frac{\varphi}{\rho} \left(\frac{y \exp\{\delta t\}}{\rho} \right)^{-\frac{1}{\varphi}} M_t^{-\frac{1}{\varphi}} \mathbb{E}_t \left[\int_t^T \left\{ \frac{M_s}{M_t} \right\}^{\frac{\varphi-1}{\varphi}} \exp \left\{ -\frac{\delta(s-t)}{\varphi} \right\} \mathbb{1}_{[M_s < \underline{\xi}_s]} ds \right] \\ &\quad - \frac{\psi\varphi}{\rho} \mathbb{E}_t \left[\int_t^T \frac{M_s}{M_t} \mathbb{1}_{[M_s < \underline{\xi}_s]} ds \right] \\ &\quad + \frac{\varphi}{\rho} \left(\frac{y \exp\{\delta t\}}{\rho \bar{\kappa}} \right)^{-\frac{1}{\varphi}} M_t^{-\frac{1}{\varphi}} \mathbb{E}_t \left[\int_t^T \left\{ \frac{M_s}{M_t} \right\}^{\frac{\varphi-1}{\varphi}} \exp \left\{ -\frac{\delta(s-t)}{\varphi} \right\} \mathbb{1}_{[\bar{\xi}_s < M_s < \bar{\xi}_s^*]} ds \right] \\ &\quad - \frac{\psi\varphi}{\rho} \mathbb{E}_t \left[\int_t^T \frac{M_s}{M_t} \mathbb{1}_{[\bar{\xi}_s < M_s < \bar{\xi}_s^*]} ds \right]. \end{aligned} \quad (\text{A12})$$

Here, $\bar{\xi}_s^* \equiv \frac{\psi^{-\varphi} \rho \bar{\kappa}}{y \exp\{\delta t\}}$. The closed-form expression for W_t^* can be determined by computing the conditional expectations. In case the investment opportunity set is constant, we find

$$\mathbb{E}_t \left[\left\{ \frac{M_s}{M_t} \right\}^{\frac{\varphi-1}{\varphi}} \exp \left\{ -\frac{\delta(s-t)}{\varphi} \right\} \mathbb{1}_{[M_s < \underline{\xi}_s]} \right] = \exp \{-C(s-t)\} \mathcal{N}[d_2(\underline{\xi}_s)], \quad (\text{A13})$$

$$\mathbb{E}_t \left[\left\{ \frac{M_s}{M_t} \right\}^{\frac{\varphi-1}{\varphi}} \exp \left\{ -\frac{\delta(s-t)}{\varphi} \right\} \mathbb{1}_{[\bar{\xi}_s < M_s < \bar{\xi}_s^*]} \right] = \exp \{ -C(s-t) \} \times (\mathcal{N}[d_2(\bar{\xi}_s^*)] - \mathcal{N}[d_2(\bar{\xi}_s)]), \quad (\text{A14})$$

$$\mathbb{E}_t \left[\frac{M_s}{M_t} \theta_s \mathbb{1}_{[\underline{\xi}_s \leq M_s \leq \bar{\xi}_s]} \right] = \theta_s \exp \{ -r(s-t) \} (\mathcal{N}[d_1(\bar{\xi}_s)] - \mathcal{N}[d_1(\underline{\xi}_s)]), \quad (\text{A15})$$

$$\mathbb{E}_t \left[\frac{M_s}{M_t} \mathbb{1}_{[\bar{\xi}_s < M_s < \bar{\xi}_s^*]} \right] = \exp \{ -r(s-t) \} (\mathcal{N}[d_1(\bar{\xi}_s^*)] - \mathcal{N}[d_1(\bar{\xi}_s)]), \quad (\text{A16})$$

$$\mathbb{E}_t \left[\frac{M_s}{M_t} \mathbb{1}_{[M_s < \underline{\xi}_s]} \right] = \exp \{ -r(s-t) \} \mathcal{N}[d_1(\underline{\xi}_s)]. \quad (\text{A17})$$

Here, \mathcal{N} is the cumulative distribution function of a standard normal random variable, and C , $d_1(x)$ and $d_2(x)$ are defined as follows:

$$C = \frac{\delta + r(\varphi - 1)}{\varphi} + \frac{1}{2} \frac{\varphi - 1}{\varphi^2} \|\lambda\|^2,$$

$$d_1(x) = \frac{1}{\|\lambda\| \sqrt{s-t}} \left[\log(x) - \log(M_t) + \left(r - \frac{1}{2} \|\lambda\|^2 \right) (s-t) \right],$$

$$d_2(x) = d_1(x) + \frac{1}{\varphi} \|\lambda\| \sqrt{s-t}.$$

Substituting the conditional expectations (A13) – (A17) into equation (A12) yields the optimal wealth. ■